

OBSTACLE PROBLEM FOR A CLASS OF PARABOLIC EQUATIONS OF GENERALIZED p -LAPLACIAN TYPE

CASIMIR LINDFORS

ABSTRACT. We study nonlinear parabolic PDEs with Orlicz-type growth conditions. The main result gives the existence of a unique solution to the obstacle problem related to these equations. To achieve this we show the boundedness of weak solutions and that a uniformly bounded sequence of weak supersolutions converges to a weak supersolution. Moreover, we prove that if the obstacle is continuous, so is the solution.

1. INTRODUCTION

In this paper we prove the existence of a unique solution to the obstacle problem related to a wide class of nonlinear parabolic equations with a merely bounded obstacle. If the obstacle is also continuous, we show that the solution inherits the same property. More specifically, we consider equations of the type

$$\partial_t u - \operatorname{div} \mathcal{A}(Du) = 0, \quad (1.1)$$

where \mathcal{A} is a C^1 vector field with $\mathcal{A}(\xi) \approx \frac{g(|\xi|)}{|\xi|} \xi$. Here $g \in C^1(\mathbb{R}_+)$ is a positive function satisfying the Orlicz-type growth condition

$$g_0 - 1 \leq \frac{sg'(s)}{g(s)} \leq g_1 - 1, \quad s > 0 \quad (1.2)$$

with $2n/(n+2) < g_0 \leq g_1 < \infty$. A function u solves the corresponding obstacle problem with the obstacle ψ if it is the smallest $\operatorname{ess\,lim\,inf}$ -regularized (see (4.7)) weak supersolution to (1.1) such that $u \geq \psi$ almost everywhere in Ω_T .

Equation (1.1) is a generalization of the widely studied evolutionary p -Laplace type equations. Indeed, when $g_0 = g_1 = p$ we have $g(s) = s^{p-1}$ up to a constant. For these equations the existence of a continuous solution to the obstacle problem with a continuous obstacle was proved in [23]. More irregular obstacles are treated in [32]. Our proofs are analogous to those in [23, 32], in fact, the main new ingredients are Theorem 3.9, which tells that a sequence of uniformly bounded weak supersolutions converges pointwise to a weak supersolution, and Theorem 4.2, stating that a nonnegative weak subsolution is locally bounded. For p -Laplace type equations the former was proved in [24], see also [31], for the latter we refer to [33, 13].

In the study of the evolutionary p -Laplacian there is a strong distinction between the degenerate ($p \geq 2$) and singular ($1 < p < 2$) cases. For the more general equations we are interested in, the main difficulty compared to the p -Laplace case arises from the fact that the equation can be both degenerate and singular. Indeed, this is possible when $g_0 < 2 < g_1$, see [4] for a concrete example. This difficulty can be seen for example in the proof of Theorem 4.2, where we merely obtain a qualitative bound for subsolutions, contrary to the p -Laplace case, see [13]. Another indication of how problematic the more general growth conditions can be is the fact that the Hölder continuity of solutions to parabolic

Date: April 12, 2016.

Key words and phrases. Degenerate/singular parabolic equations; General growth conditions; Obstacle problem.

equations with only measurable coefficients is still an open problem. Purely degenerate and purely singular cases have been treated in [21] and [22], respectively, see also [20]. Operators satisfying the more general growth conditions were first systematically studied in [27]. Further developments have been made in [12, 15, 3] in the elliptic setting and, in addition to the ones mentioned above, the parabolic case has been studied in [28, 29, 7]. The variational counterpart has been treated in [14, 18, 10, 11].

Motivation to study equations with more general growth comes from many physical phenomena that cannot be modeled sufficiently accurately using polynomial growth. For example, the stationary, irrotational flow of a compressible fluid can be modeled using an equation of the type

$$\operatorname{div} [\rho(|Du|^2)Du] = 0,$$

where Du is the velocity field of the flow and $|Du| =: q$ the speed of the flow. In this context one introduces the *Mach number*

$$M^2 \equiv [M(q)]^2 := -\frac{2q^2}{\rho(q^2)}\rho'(q^2)$$

(note that we must have $\rho' < 0$). The general theory asserts that a point is *elliptic* if $M < 1$ and in this case the flow is *subsonic*, while if $M > 1$ the point is *hyperbolic* and the flow there is *supersonic*. If $M = 1$ the flow is called *sonic*. In our context, where $g(s) = \rho(s^2)s$, we compute the Orlicz ratio $sg'(s)/g(s) = 1 - M(s)^2$. Thus, if we know that the flow maintains a controlled, small speed q , then the problem falls in the class of operators we consider. For further details see for instance [5, 16, 17].

Obstacle problems are a widely studied topic in the theory of partial differential equations. This is due to the fact that obstacle problems have numerous applications in several different areas of science, including physics, chemistry, biology, and even finance. Moreover, obstacle problems have turned out to be a fundamental tool in potential theory. The regularity of solutions to obstacle problems and the related free boundary problems is a classical topic in partial differential equations; for this we refer to [2, 8]. For more recent advances in the parabolic setting see [23, 32, 30, 26]. The elliptic case has been treated comprehensively in [19].

The paper is organized as follows. In Section 2 we introduce the basic properties of the function g and some useful results for the related Orlicz spaces. In Section 3 we state known results for weak solutions to (1.1) and prove the convergence result Theorem 3.9. The boundedness of solutions is established in Section 4, and for this we prove an *a priori* result (Lemma 4.1) which we find interesting in its own right. Moreover, we show that weak supersolutions to (1.1) are lower semicontinuous (Theorem 4.6). Finally, in Section 5 we prove the existence result for the obstacle problem with a bounded obstacle (Theorem 5.2), and show that if the obstacle is continuous, so is the solution (Theorem 5.14).

2. PRELIMINARIES

Let $n \geq 2$ and $\Omega_T = \Omega \times (0, T) \subset \mathbb{R}^n \times \mathbb{R}$, where Ω is a bounded domain. Consider the equation

$$\partial_t u - \operatorname{div} \mathcal{A}(Du) = 0 \quad \text{in} \quad \Omega_T \quad (2.1)$$

where $\mathcal{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a C^1 vector field satisfying

$$\begin{cases} \langle D\mathcal{A}(\xi)\zeta, \zeta \rangle \geq \nu \frac{g(|\xi|)}{|\xi|} |\zeta|^2 \\ |D\mathcal{A}(\xi)| \leq L \frac{g(|\xi|)}{|\xi|} \end{cases}, \quad (2.2)$$

for every $\xi \in \mathbb{R}^n \setminus \{0\}, \zeta \in \mathbb{R}^n$ and with structural constants $0 < \nu \leq 1 \leq L$. We may assume without loss of generality that $\mathcal{A}(0) = 0$ by replacing $\mathcal{A}(\xi)$ with $\mathcal{A}(\xi) - \mathcal{A}(0)$.

The function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is assumed to be C^1 -regular and to satisfy (1.2). Moreover, without loss of generality we may assume that

$$\int_0^1 g(\rho) d\rho = 1 \quad (2.3)$$

by scaling g with a suitable constant and changing the structural constants accordingly.

Remark 2.1. Our results hold for a wider class of operators \mathcal{A} , which allow the presence of a function g that is not C^1 but merely Lipschitz. Indeed, we may consider Lipschitz functions $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying (1.2) almost everywhere and vector fields $\mathcal{A} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ in $W^{1,\infty}(\mathbb{R}^n)$ satisfying the monotonicity and Lipschitz assumptions

$$\begin{cases} \langle \mathcal{A}(\xi_1) - \mathcal{A}(\xi_2), \xi_1 - \xi_2 \rangle \geq \nu \frac{g(|\xi_1| + |\xi_2|)}{|\xi_1| + |\xi_2|} |\xi_1 - \xi_2|^2, \\ |\mathcal{A}(\xi_1) - \mathcal{A}(\xi_2)| \leq L \frac{g(|\xi_1| + |\xi_2|)}{|\xi_1| + |\xi_2|} |\xi_1 - \xi_2|, \end{cases} \quad (2.4)$$

for every $\xi_1, \xi_2 \in \mathbb{R}^n$ such that $|\xi_1| + |\xi_2| \neq 0$ and for some $0 < \nu \leq 1 \leq L$.

2.1. Notation. We denote by c a general constant *always larger than or equal to one*, possibly varying from line to line; relevant dependencies on parameters will be emphasized using parentheses, i.e., $c \equiv c(n, g_0, g_1)$ means that c depends on n, g_0, g_1 .

We denote by

$$B_R(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < R\}$$

the open ball with center x_0 and radius $R > 0$; when clear from the context or otherwise not important, we shall omit the center. Unless otherwise explicitly stated, different balls and cylinders in the same context will have the same center. The parabolic boundary of a cylindrical domain $\mathcal{K} = \mathcal{D} \times (t_1, t_2) \subset \mathbb{R}^n \times \mathbb{R}$ is defined as

$$\partial_p \mathcal{K} := (\overline{\mathcal{D}} \times \{t_1\}) \cup (\partial \mathcal{D} \times (t_1, t_2)).$$

Naturally, the parabolic closure of \mathcal{K} is then $\overline{\mathcal{K}}^p := \mathcal{K} \cup \partial_p \mathcal{K}$. Accordingly with the customary use in the parabolic setting, when considering a sub-cylinder \mathcal{K} (as above) compactly contained in Ω_T , we shall mean that $\mathcal{D} \Subset \Omega$ and $0 < t_1 < t_2 \leq T$; we will write in this case $\mathcal{K} \Subset \Omega_T$.

With $\mathcal{B} \subset \mathbb{R}^\ell$ being a measurable set, $\chi_{\mathcal{B}}$ denotes its characteristic function. If furthermore \mathcal{B} has positive and finite measure and $f : \mathcal{B} \rightarrow \mathbb{R}^k$ is a measurable map, we shall denote by

$$\oint_{\mathcal{B}} f(y) dy := \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} f(y) dy$$

the integral average of f over \mathcal{B} . We shall also as usual denote

$$\operatorname{ess\,osc}_{\mathcal{B}} f := \operatorname{ess\,sup}_{\mathcal{B}} f - \operatorname{ess\,inf}_{\mathcal{B}} f.$$

By q^* we denote the Sobolev conjugate exponent of $1 \leq q < n$, i.e.,

$$q^* := \frac{nq}{n - q} \quad (2.5)$$

With s being a real number, we denote $s_+ := \max\{s, 0\}$ and $s_- := \max\{-s, 0\}$. Finally, $\mathbb{R}_+ := [0, \infty)$, \mathbb{N} is the set $\{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.2. Properties of g and basic inequalities. We begin by collecting useful properties of the function g and some basic inequalities that will be needed later. For proofs see for example [1] and [15].

First, observe that g is strictly increasing and satisfies $g(0) = 0$ and $\lim_{s \rightarrow \infty} g(s) = \infty$. Define the function $G : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as

$$G(s) := \int_0^s g(r) dr \quad (2.6)$$

and its Young complement as

$$\tilde{G}(s) := \sup_{r \geq 0} (sr - G(r)).$$

The functions G and \tilde{G} are strictly increasing, strictly convex, and map zero to zero, in particular, they are Young functions. Moreover, they both satisfy an Orlicz-type condition, namely for $s > 0$

$$g_0 \leq \frac{sG'(s)}{G(s)} \leq g_1 \quad \text{and} \quad \frac{g_1}{g_1 - 1} \leq \frac{s\tilde{G}'(s)}{\tilde{G}(s)} \leq \frac{g_0}{g_0 - 1}. \quad (2.7)$$

For g we have the so-called Δ_2 -condition

$$\min\{\alpha^{g_0-1}, \alpha^{g_1-1}\}g(s) \leq g(\alpha s) \leq \max\{\alpha^{g_0-1}, \alpha^{g_1-1}\}g(s), \quad (2.8)$$

and corresponding inequalities hold also for G and \tilde{G} . Moreover, G satisfies the triangle inequality modulo a constant

$$G(s+r) \leq 2^{g_1}(G(s) + G(r)), \quad s, r \geq 0, \quad (2.9)$$

and the following important inequality

$$\tilde{G}\left(\frac{G(s)}{s}\right) \leq G(s), \quad s > 0. \quad (2.10)$$

Finally, for $\varepsilon \in (0, 1]$ we have the Young's inequality with ε

$$sr \leq \varepsilon G(s) + \varepsilon^{-\frac{1}{g_0-1}} \tilde{G}(r), \quad s, r \geq 0.$$

By writing

$$\mathcal{A}(\xi) = \int_0^1 D\mathcal{A}(s\xi)\xi \, ds$$

we easily see that assumptions (2.2) imply

$$\begin{cases} \langle \mathcal{A}(\xi), \xi \rangle \geq \tilde{\nu} G(|\xi|) \\ |\mathcal{A}(\xi)| \leq \tilde{L} \frac{G(|\xi|)}{|\xi|} \end{cases} \quad (2.11)$$

for $\xi \in \mathbb{R}^n$ with $\tilde{\nu} := \frac{g_0}{g_1-1}\nu$ and $\tilde{L} := \frac{g_1}{g_0-1}L$, and in the case $G(s) = s^p$ these are precisely the commonly used assumptions in the study of p -Laplace type equations.

Lemma 2.2. (Strict monotonicity) *There exists a constant $c \equiv c(g_0, g_1, \nu)$ such that*

$$\langle \mathcal{A}(\xi_1) - \mathcal{A}(\xi_2), \xi_1 - \xi_2 \rangle \geq cg'(|\xi_1| + |\xi_2|)|\xi_1 - \xi_2|^2$$

for every $\xi_1, \xi_2 \in \mathbb{R}^n$.

Remark 2.3. Note that since g is increasing, Lemma 2.2 implies

$$\langle \mathcal{A}(\xi_1) - \mathcal{A}(\xi_2), \xi_1 - \xi_2 \rangle \geq 0$$

for every $\xi_1, \xi_2 \in \mathbb{R}^n$.

Define the natural quantity $V_g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$V_g(\xi) = \left(\frac{g(|\xi|)}{|\xi|} \right)^{\frac{1}{2}} \xi$$

when $\xi \neq 0$ and set $V_g(0) = 0$. Clearly V_g is continuous and, moreover, has a continuous inverse by the inverse function theorem.

Lemma 2.4. *There exists a constant $c \equiv c(g_0, g_1)$ such that*

$$|V_g(\xi_1) - V_g(\xi_2)|^2 \leq cg'(|\xi_1| + |\xi_2|)|\xi_1 - \xi_2|^2$$

for every $\xi_1, \xi_2 \in \mathbb{R}^n$.

2.3. Orlicz spaces. Let G be as in (2.6). A measurable function $u : \Omega \rightarrow \mathbb{R}$ belongs to the Orlicz space $L^G(\Omega)$ if it satisfies

$$\int_{\Omega} G(|u|) dx < \infty.$$

The space $L^G(\Omega)$ is a vector space, since G satisfies the Δ_2 -condition, and it can be shown to be a Banach space if endowed with the Luxemburg norm

$$\|u\|_{L^G(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} G\left(\frac{|u|}{\lambda}\right) dx \leq 1 \right\}.$$

A function u belongs to $L^G_{\text{loc}}(\Omega)$, if $u \in L^G(\Omega')$ for every $\Omega' \Subset \Omega$. If also the weak gradient of u belongs to $L^G(\Omega)$, we say that $u \in W^{1,G}(\Omega)$. The corresponding space with zero boundary values, denoted $W^{1,G}_0(\Omega)$, is the completion of $C^\infty_c(\Omega)$ under the norm

$$\|u\|_{W^{1,G}(\Omega)} := \|u\|_{L^G(\Omega)} + \|Du\|_{L^G(\Omega)}.$$

We denote by $V^G(\Omega_T)$ the space of functions $u \in L^G(\Omega_T) \cap L^1(0, T; W^{1,1}(\Omega))$ for which also the weak spatial gradient Du belongs to $L^G(\Omega_T)$. The space $V^G(\Omega_T)$ is also a Banach space with the norm

$$\|u\|_{V^G(\Omega_T)} := \|u\|_{L^G(\Omega_T)} + \|Du\|_{L^G(\Omega_T)}.$$

Moreover, we denote by $V^{1,G}_0(\Omega_T)$ the space of functions $u \in V^G(\Omega_T)$ for which $u(\cdot, t)$ belongs to $W^{1,G}_0(\Omega)$ for almost every $t \in (0, T)$, while the localized version $V^{1,G}_{\text{loc}}(\Omega_T)$ is defined, as above, in the customary way. We also use the shorthand notation

$$V^{2,G}(\Omega_T) := L^\infty(0, T; L^2(\Omega)) \cap V^G(\Omega_T)$$

and similarly for the localized and the zero trace versions. More on Orlicz spaces can be found, for example, in [1].

We gather here some useful results for Orlicz space functions.

Lemma 2.5. (Hölder's inequality) *Let $u \in L^G(\Omega_T)$ and $v \in L^{\tilde{G}}(\Omega_T)$. Then $uv \in L^1(\Omega_T)$ and we have*

$$\int_{\Omega_T} |u| |v| dx dt \leq 2 \|u\|_{L^G(\Omega_T)} \|v\|_{L^{\tilde{G}}(\Omega_T)}.$$

Proof. See for example [1]. □

Lemma 2.6. *The inequality*

$$\|\chi_E\|_{L^G(\Omega_T)} \leq \max \left\{ |E|^{\frac{1}{g_1}}, |E|^{\frac{1}{g_0}} \right\}.$$

holds for every $E \subset \Omega_T$.

Proof. We may assume $|E| > 0$, since the claim trivially holds if $|E| = 0$. It is easy to show that $G^{-1}(s) \geq \min \left\{ s^{\frac{1}{g_1}}, s^{\frac{1}{g_0}} \right\}$ for every $s \geq 0$. Now, since

$$\int_{\Omega_T} G \left(G^{-1} \left(\frac{1}{|E|} \right) \chi_E \right) dx dt = 1,$$

we have

$$\|\chi_E\|_{L^G(\Omega_T)} = \frac{1}{G^{-1}(1/|E|)} \leq \max \left\{ |E|^{\frac{1}{g_1}}, |E|^{\frac{1}{g_0}} \right\}. \quad \square$$

Lemma 2.7. (Poincaré's inequality) *Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Then*

$$\int_{\Omega} G \left(\frac{|u|}{\text{diam}(\Omega)} \right) dx \leq \int_{\Omega} G(|Du|) dx$$

for every $u \in W^{1,G}_0(\Omega)$.

Proof. See Lemma 2.2 in [27]. \square

Lemma 2.8. (Parabolic Sobolev's inequality) *Let $1 \leq q < \min\{n, g_0\}$ and $B_R \times \Gamma \subset \mathbb{R}^{n+1}$. Then there exists a constant $c \equiv c(n, g_1, q)$ such that*

$$\begin{aligned} \int_{B_R \times \Gamma} G\left(\frac{|u|}{R}\right)^{1/q} |u|^{2(1-1/q^*)} dx dt \\ \leq c \operatorname{ess\,sup}_{\Gamma} \left(\int_{B_R} |u|^2 dx \right)^{1-1/q^*} \left(\int_{B_R \times \Gamma} G(|Du|) dx dt \right)^{1/q} \end{aligned}$$

for every $u \in V_0^{2,G}(B_R \times \Gamma)$.

Proof. Set $H(s) := G(s^{1/q})$ and $F(s) := \tilde{H}(s^q)^{1/q}$. Simple calculations show that

$$\frac{g_0}{q} \leq \frac{sH'(s)}{H(s)} \leq \frac{g_1}{q} \quad \text{and} \quad \frac{g_1}{g_1 - q} \leq \frac{sF'(s)}{F(s)} \leq \frac{g_0}{g_0 - q}$$

for every $s > 0$. Moreover, the elementary inequality $a^q - b^q \geq (a - b)^q$, $a \geq b \geq 0$, yields

$$\begin{aligned} F(s) &= \left(\sup_{r>0} (s^q r - G(r^{1/q})) \right)^{1/q} = \sup_{r>0} ((sr)^q - G(r))^{1/q} \\ &= \sup_{\substack{r>0 \\ (sr)^q \geq G(r)}} ((sr)^q - G(r))^{1/q} \geq \sup_{\substack{r>0 \\ (sr)^q \geq G(r)}} (sr - G(r)^{1/q}) \\ &= \sup_{r>0} (sr - G(r)^{1/q}), \end{aligned}$$

and thus $\tilde{F}(s) \leq G(s)^{1/q}$ due to the fact that for Young functions A and B , $A \leq B$ implies $\tilde{A} \leq \tilde{B}$. Now for almost every $t \in \Gamma$ we have

$$\begin{aligned} \left| D\tilde{F}\left(\frac{|u(\cdot, t)|}{R}\right) \right|^q &= \tilde{F}'\left(\frac{|u(\cdot, t)|}{R}\right)^q \left(\frac{|Du(\cdot, t)|}{R}\right)^q \\ &\leq \frac{c(g_1, q)}{R^q} \left(\frac{\tilde{F}(|u(\cdot, t)|/R)}{|u(\cdot, t)|/R} \right)^q |Du(\cdot, t)|^q \\ &\leq \frac{c(g_1, q)}{R^q} \left[\varepsilon \tilde{H}\left(\left(\frac{\tilde{F}(|u(\cdot, t)|/R)}{|u(\cdot, t)|/R}\right)^q\right) + \varepsilon^{1-g_1/q} H(|Du(\cdot, t)|^q) \right] \\ &= \frac{c(g_1, q)}{R^q} \left[\varepsilon F\left(\frac{\tilde{F}(|u(\cdot, t)|/R)}{|u(\cdot, t)|/R}\right)^q + \varepsilon^{1-g_1/q} G(|Du(\cdot, t)|) \right] \\ &\leq \frac{c(g_1, q)}{R^q} \left[\varepsilon \tilde{F}\left(\frac{|u(\cdot, t)|}{R}\right)^q + \varepsilon^{1-g_1/q} G(|Du(\cdot, t)|) \right] \end{aligned}$$

by Young's inequality with $\varepsilon \in (0, 1)$, the definitions of F and H , and (2.10). This implies $\tilde{F}(|u(\cdot, t)|/R) \in W_0^{1,q}(B_R)$ for almost every $t \in \Gamma$, since $u \in V_0^G(B_R \times \Gamma)$. Therefore we may apply the elliptic Sobolev's inequality to obtain

$$\begin{aligned} \left(\int_{B_R} \tilde{F}\left(\frac{|u(\cdot, t)|}{R}\right)^{q^*} dx \right)^{q/q^*} &\leq c(n, q) R^q \int_{B_R} \left| D\tilde{F}\left(\frac{|u(\cdot, t)|}{R}\right) \right|^q dx \\ &\leq c(n, g_1, q) \int_{B_R} \left[\varepsilon \tilde{F}\left(\frac{|u(\cdot, t)|}{R}\right)^q + \varepsilon^{1-g_1/q} G(|Du(\cdot, t)|) \right] dx \\ &\leq \frac{1}{2} \left(\int_{B_R} \tilde{F}\left(\frac{|u(\cdot, t)|}{R}\right)^{q^*} dx \right)^{q/q^*} + c(n, g_1, q) \int_{B_R} G(|Du(\cdot, t)|) dx, \end{aligned}$$

where we also used Hölder's inequality and chose $\varepsilon \equiv \varepsilon(n, g_1, q)$ small enough. Hence

$$\left(\int_{B_R} \tilde{F} \left(\frac{|u(\cdot, t)|}{R} \right)^{q^*} dx \right)^{1/q^*} \leq c \left(\int_{B_R} G(|Du(\cdot, t)|) dx \right)^{1/q} \quad (2.12)$$

for almost every $t \in \Gamma$, where $c \equiv c(n, g_1, q)$.

Using another elementary inequality $a^q - 2^q b^q \leq 2^q(a - b)^q$, $a \geq b \geq 0$, gives

$$\begin{aligned} F(s) &= \sup_{r>0} ((sr)^q - G(r))^{1/q} = \sup_{\substack{r>0 \\ (sr)^q \geq G(r)}} \left((sr)^q - 2^q \left(\frac{1}{2} G(r)^{1/q} \right)^q \right)^{1/q} \\ &\leq 2 \sup_{\substack{r>0 \\ (2sr)^q \geq G(r)}} \left(sr - \frac{1}{2} G(r)^{1/q} \right) = \sup_{r>0} \left(sr - G\left(\frac{r}{2}\right)^{1/q} \right), \end{aligned}$$

and therefore

$$\tilde{F}(s) \geq G\left(\frac{s}{2}\right)^{1/q} \geq 2^{-g_1/q} G(s)^{1/q}.$$

By combining this with Hölder's inequality and (2.12) we finally obtain

$$\begin{aligned} \int_{B_R \times \Gamma} G\left(\frac{|u|}{R}\right)^{1/q} |u|^{2(1-1/q^*)} dx dt &\leq c \int_{B_R \times \Gamma} \tilde{F}\left(\frac{|u|}{R}\right) |u|^{2(1-1/q^*)} dx dt \\ &\leq c \int_{\Gamma} \left(\int_{B_R} \tilde{F}\left(\frac{|u|}{R}\right)^{q^*} dx \right)^{1/q^*} \left(\int_{B_R} |u|^2 dx \right)^{1-1/q^*} dt \\ &\leq c \operatorname{ess\,sup}_{\Gamma} \left(\int_{B_R} |u|^2 dx \right)^{1-1/q^*} \left(\int_{B_R \times \Gamma} G(|Du|) dx dt \right)^{1/q}. \quad \square \end{aligned}$$

Remark 2.9. Since $g_0 > 2n/(n+2)$ we may take $q = 2n/(n+2)$ in the previous Lemma, which yields

$$\begin{aligned} \int_{B_R \times \Gamma} G\left(\frac{|u|}{R}\right)^{1/2+1/n} |u| dx dt \\ \leq c \operatorname{ess\,sup}_{\Gamma} \left(\int_{B_R} |u|^2 dx \right)^{1/2} \left(\int_{B_R \times \Gamma} G(|Du|) dx dt \right)^{1/2+1/n} \end{aligned}$$

for every $u \in V_0^{2,G}(B_R \times \Gamma)$.

3. USEFUL RESULTS FOR SOLUTIONS

In this section we collect and partly prove various results for weak solutions to (1.1) that are standard for the evolutionary p -Laplace equation. The main result of the section is Theorem 3.9, which states that the limit of a uniformly bounded sequence of weak supersolutions is also a weak supersolution.

We begin with the definition of weak solutions.

Definition 3.1. A function u is a *weak solution* in Ω_T , if $u \in V_{\text{loc}}^{2,G}(\Omega_T)$ and it satisfies

$$- \int_{\Omega_T} u \partial_t \eta dx dt + \int_{\Omega_T} \mathcal{A}(Du) \cdot D\eta dx dt = 0 \quad (3.1)$$

for every $\eta \in C_c^\infty(\Omega_T)$. If instead of equality we have \geq (\leq) for every nonnegative $\eta \in C_c^\infty(\Omega_T)$, we say that u is a *weak supersolution* (*subsolution*) in Ω_T .

Remark 3.2. If u is a weak supersolution, it is easy to see that $-u$ is a weak subsolution to the same equation with $\mathcal{A}(Du)$ replaced by $-\mathcal{A}(-Du)$. However, since the latter equation also satisfies the structural conditions (2.2), we may assume without loss of generality that $\mathcal{A}(\xi) = -\mathcal{A}(-\xi)$. Therefore, if u is a weak supersolution, then $-u$ is a weak subsolution.

The following Caccioppoli inequality is proven in [4].

Lemma 3.3 (Caccioppoli inequality). *Let $\mathcal{K} := \mathcal{D} \times (t_1, t_2) \Subset \Omega_T$ and let u be a weak solution in \mathcal{K} . Then there exists a constant $c \equiv c(g_0, g_1, \nu, L)$ such that*

$$\begin{aligned} & \operatorname{ess\,sup}_{(t_1, t_2)} \int_{\mathcal{D}} (u - k)_{\pm}^2 \varphi^{g_1} dx + \int_{\mathcal{K}} G(|D(u - k)_{\pm}|) \varphi^{g_1} dx dt \\ & \leq \int_{\mathcal{D}} [(u - k)_{\pm}^2 \varphi^{g_1}](\cdot, t_1) dx + c \int_{\mathcal{K}} [G(|D\varphi|(u - k)_{\pm}) + (u - k)_{\pm}^2 |\partial_t \varphi|] dx dt \end{aligned}$$

for any $k \in \mathbb{R}$ and for every $\varphi \in W^{1,\infty}(\mathcal{K})$ vanishing in a neighborhood of $\partial\mathcal{D} \times (t_1, t_2)$ and with $0 \leq \varphi \leq 1$. The same inequality but only with the “+” sign holds for weak subsolutions.

For the following comparison principle we add the extra assumption that the functions in question are continuous. This weaker version is sufficient for our purposes. Again, the proof can be found in [4].

Proposition 3.4. (Comparison principle) *Let $\mathcal{K} := \mathcal{D} \times (t_1, t_2) \subset \Omega_T$ and let $u \in C^0(\overline{\mathcal{K}}^p)$ be a weak subsolution and $v \in C^0(\overline{\mathcal{K}}^p)$ a weak supersolution in \mathcal{K} . If $u \leq v$ on $\partial_p \mathcal{K}$, then $u \leq v$ in $\overline{\mathcal{K}}^p$.*

We obtain the maximum principle as an easy corollary.

Corollary 3.5. (Maximum principle) *Let $\mathcal{K} \subset \Omega_T$ and let $u \in C^0(\overline{\mathcal{K}}^p)$ be a weak solution in \mathcal{K} . Then*

$$\inf_{\partial_p \mathcal{K}} u \leq u \leq \sup_{\partial_p \mathcal{K}} u \quad (3.2)$$

in $\overline{\mathcal{K}}^p$ and, moreover,

$$\sup_{\overline{\mathcal{K}}^p} |u| = \sup_{\partial_p \mathcal{K}} |u|. \quad (3.3)$$

The next pasting lemma states that if we replace a supersolution by a smaller supersolution in some part of the cylinder such that they coincide on the boundary, the resulting function is still a supersolution. Again, we assume that the functions are continuous. The proof follows ideas used in [24].

Lemma 3.6. *Let $Q_1 := K_1 \times (t_1, \tau_1)$, $Q_2 := K_2 \times (t_2, \tau_2) \subset \Omega_T$ be such that $\tau_1 \leq \tau_2$, and let $v_1 \in C^0(\overline{Q}_1^p)$ and $v_2 \in C^0(\overline{Q}_2^p)$ be weak supersolutions (subsolutions) in Q_1 and Q_2 , respectively. If $v_1 \geq v_2$ ($v_1 \leq v_2$) in $Q_1 \cap Q_2$ and $v_1 = v_2$ on $Q_1 \cap \partial_p Q_2$, then*

$$v = \begin{cases} v_1 & \text{in } Q_1 \setminus Q_2 \\ v_2 & \text{in } Q_1 \cap Q_2 \end{cases}$$

is a weak supersolution (subsolution) in Q_1 .

Proof. Fix a nonnegative $\varphi \in C_c^\infty(Q_1)$, $\varepsilon > 0$, and define

$$w_\varepsilon := \begin{cases} 1, & v_1 \geq v_2 + 2\varepsilon \\ \frac{v_1 - v_2 - \varepsilon}{\varepsilon}, & v_2 + \varepsilon \leq v_1 < v_2 + 2\varepsilon \\ 0, & v_1 < v_2 + \varepsilon \end{cases}.$$

in $Q_1 \cap Q_2$ and $w_\varepsilon = 0$ in $Q_1 \setminus Q_2$. The test functions

$$\eta_1 = (1 - w_\varepsilon)\varphi \quad \text{and} \quad \eta_2 = w_\varepsilon\varphi$$

have compact support in Q_1 and $Q_1 \cap Q_2$, respectively, and can thus be regularized using mollification; we shall proceed formally. Now, assuming v_1 and v_2 are weak supersolutions, summing up their respective weak formulations we obtain

$$\begin{aligned}
0 &\leq - \int_{Q_1} v_1 \partial_t ((1 - w_\varepsilon) \varphi) dx dt + \int_{Q_1} \mathcal{A}(Dv_1) \cdot D((1 - w_\varepsilon) \varphi) dx dt \\
&\quad - \int_{Q_1} v_2 \partial_t (w_\varepsilon \varphi) dx dt + \int_{Q_1} \mathcal{A}(Dv_2) \cdot D(w_\varepsilon \varphi) dx dt \\
&= - \int_{Q_1} (v_1(1 - w_\varepsilon) + v_2 w_\varepsilon) \partial_t \varphi dx dt + \int_{Q_1} (v_1 - v_2) \partial_t w_\varepsilon \varphi dx dt \\
&\quad + \int_{Q_1} (\mathcal{A}(Dv_1)(1 - w_\varepsilon) + \mathcal{A}(Dv_2)w_\varepsilon) \cdot D\varphi dx dt \\
&\quad - \int_{Q_1} (\mathcal{A}(Dv_1) - \mathcal{A}(Dv_2)) \cdot Dw_\varepsilon \varphi dx dt.
\end{aligned}$$

The monotonicity of \mathcal{A} yields

$$\begin{aligned}
&\int_{Q_1} (\mathcal{A}(Dv_1) - \mathcal{A}(Dv_2)) \cdot Dw_\varepsilon \varphi dx dt \\
&= \frac{1}{\varepsilon} \int_{Q_1} (\mathcal{A}(Dv_1) - \mathcal{A}(Dv_2)) \cdot (Dv_1 - Dv_2) \chi_{\{v_2 + \varepsilon \leq v_1 < v_2 + 2\varepsilon\}} \varphi dx dt \geq 0,
\end{aligned}$$

and by integration by parts we get

$$\begin{aligned}
&\int_{Q_1} (v_1 - v_2) \partial_t w_\varepsilon \varphi dx dt = \varepsilon \int_{Q_1} (1 + w_\varepsilon) \partial_t w_\varepsilon \varphi dx dt \\
&= -\frac{\varepsilon}{2} \int_{Q_1} (1 + w_\varepsilon)^2 \partial_t \varphi dx dt \leq 2\varepsilon \|\partial_t \varphi\|_{L^\infty(Q_1)} |Q_1|,
\end{aligned}$$

since $w_\varepsilon \leq 1$. Therefore,

$$\begin{aligned}
0 &\leq - \int_{Q_1} (v_1(1 - w_\varepsilon) + v_2 w_\varepsilon) \partial_t \varphi dx dt + 2\varepsilon \|\partial_t \varphi\|_{L^\infty(Q_1)} |Q_1| \\
&\quad + \int_{Q_1} (\mathcal{A}(Dv_1)(1 - w_\varepsilon) + \mathcal{A}(Dv_2)w_\varepsilon) \cdot D\varphi dx dt.
\end{aligned}$$

Since $w_\varepsilon = 0$ in $Q_1 \setminus Q_2$ and $v_1 \geq v_2$ in $Q_1 \cap Q_2$, we have

$$|w_\varepsilon - \chi_{Q_1 \cap Q_2}| \leq \chi_{Q_1 \cap Q_2 \cap \{v_2 \leq v_1 < v_2 + 2\varepsilon\}} \rightarrow 0$$

as $\varepsilon \rightarrow 0$, and thus, we obtain

$$\begin{aligned}
0 &\leq - \lim_{\varepsilon \rightarrow 0} \int_{Q_1} (v_1(1 - w_\varepsilon) + v_2 w_\varepsilon) \partial_t \varphi dx dt \\
&\quad + \lim_{\varepsilon \rightarrow 0} \int_{Q_1} (\mathcal{A}(Dv_1)(1 - w_\varepsilon) + \mathcal{A}(Dv_2)w_\varepsilon) \cdot D\varphi dx dt \\
&= - \int_{Q_1} (v_1 \chi_{Q_1 \setminus Q_2} + v_2 \chi_{Q_1 \cap Q_2}) \partial_t \varphi dx dt \\
&\quad + \int_{Q_1} (\mathcal{A}(Dv_1) \chi_{Q_1 \setminus Q_2} + \mathcal{A}(Dv_2) \chi_{Q_1 \cap Q_2}) \cdot D\varphi dx dt \\
&= - \int_{Q_1} v \partial_t \varphi dx dt + \int_{Q_1} \mathcal{A}(Dv) \cdot D\varphi dx dt,
\end{aligned}$$

showing that v is a weak supersolution in Q_1 .

If v_1 and v_2 are assumed to be weak subsolutions such that $v_1 \leq v_2$, then by applying the above reasoning to $-v_1$ and $-v_2$ we see that $-v$ is a weak supersolution, and the result follows. \square

The following lemma can be proved in a very similar fashion as the previous one.

Lemma 3.7. *Let u and v be weak supersolutions (subsolutions) in $Q \subset \Omega_T$. Then also $\min\{u, v\}$ is a weak supersolution ($\max\{u, v\}$ is a weak subsolution) in Q .*

3.1. Convergence properties of supersolutions. We end the section by proving an important convergence result that is crucial in proving the existence of a solution to the obstacle problem. For this we need the following lemma. The proof follows the guidelines of Theorem 6 in [31], see also [24, 6].

Lemma 3.8. *Let $Q \subset \Omega_T$, $M \geq 1$, and let $(u_i)_{i=1}^\infty$ be a sequence of weak supersolutions in Q such that $|u_i| \leq M$ for every $i \in \mathbb{N}$ and $u_i \rightarrow u$ almost everywhere in Q . Then $Du_i \rightarrow Du$ almost everywhere in Q .*

Proof. Let $Q' \Subset Q$ and choose \tilde{Q} such that $Q' \Subset \tilde{Q} \Subset Q$. Let $\varphi \in C_c^\infty(Q)$, $\tilde{\varphi} \in C_c^\infty(\tilde{Q})$ be such that $0 \leq \varphi, \tilde{\varphi} \leq 1$, $\varphi = 1$ in \tilde{Q} , $\tilde{\varphi} = 1$ in Q' , and

$$\|\partial_t \varphi\|_{L^\infty(Q)}, \|\partial_t \tilde{\varphi}\|_{L^\infty(Q)}, \|D\varphi\|_{L^\infty(Q)}, \|D\tilde{\varphi}\|_{L^\infty(Q)} \leq C$$

for some $C \geq 1$. Applying the Caccioppoli estimate, Lemma 3.3, to the nonnegative weak subsolution $M - u_i$ (with $k = 0$) gives

$$\begin{aligned} \int_{\tilde{Q}} G(|Du_i|) dx dt &\leq c \int_Q (M - u_i)^2 |\partial_t \varphi| dx dt \\ &\quad + c \int_Q G(|D\varphi|(M - u_i)) dx dt \\ &\leq c(M^2 C + M^{g_1} C^{g_1})|Q| =: M_1. \end{aligned}$$

Thus, the sequence $(Du_i)_{i=1}^\infty$ is uniformly bounded in $L^G(\tilde{Q})$. Moreover, the sequence $(\mathcal{A}(Du_i))_{i=1}^\infty$ is uniformly bounded in $L^{\tilde{G}}(\tilde{Q})$, since

$$\begin{aligned} \int_{\tilde{Q}} \tilde{G}(|\mathcal{A}(Du_i)|) dx dt &\leq \int_{\tilde{Q}} \tilde{G}\left(\frac{g_1 L}{g_0 - 1} \frac{G(|Du_i|)}{|Du_i|}\right) dx dt \\ &\leq c \int_{\tilde{Q}} G(|Du_i|) dx dt \leq c M_1 =: M_2. \end{aligned}$$

by (2.2)₂ and (2.10). Assuming without loss of generality that $M_2 \geq 1$ it is then easy to see that also

$$\|\mathcal{A}(Du_i)\|_{L^{\tilde{G}}(\tilde{Q})} \leq M_2.$$

Denote for $j, k \in \mathbb{N}$

$$w_{jk} := \begin{cases} \delta, & u_j - u_k > \delta \\ u_j - u_k, & |u_j - u_k| \leq \delta \\ -\delta, & u_j - u_k < -\delta \end{cases},$$

where $\delta > 0$. Choose

$$\eta_j = (\delta - w_{jk})\tilde{\varphi} \quad \text{and} \quad \eta_k = (\delta + w_{jk})\tilde{\varphi}$$

as the test functions in (3.1) for the weak supersolutions u_j and u_k . Observe that η_j and η_k are nonnegative. These formal choices can be justified by standard regularization methods.

Summing up the weak formulations yields

$$\begin{aligned}
0 &\leq - \int_Q u_j \partial_t \eta_j \, dx \, dt + \int_Q \mathcal{A}(Du_j) \cdot D\eta_j \, dx \, dt \\
&\quad - \int_Q u_k \partial_t \eta_k \, dx \, dt + \int_Q \mathcal{A}(Du_k) \cdot D\eta_k \, dx \, dt \\
&= \int_{\tilde{Q}} (u_j - u_k) \partial_t w_{jk} \tilde{\varphi} \, dx \, dt + \int_{\tilde{Q}} (u_j - u_k) w_{jk} \partial_t \tilde{\varphi} \, dx \, dt \\
&\quad - \delta \int_{\tilde{Q}} (u_j + u_k) \partial_t \tilde{\varphi} \, dx \, dt - \int_{\tilde{Q}} (\mathcal{A}(Du_j) - \mathcal{A}(Du_k)) \cdot Dw_{jk} \tilde{\varphi} \, dx \, dt \\
&\quad - \int_{\tilde{Q}} (\mathcal{A}(Du_j) - \mathcal{A}(Du_k)) \cdot D\tilde{\varphi} w_{jk} \, dx \, dt \\
&\quad + \delta \int_{\tilde{Q}} (\mathcal{A}(Du_j) + \mathcal{A}(Du_k)) \cdot D\tilde{\varphi} \, dx \, dt.
\end{aligned} \tag{3.4}$$

Integration by parts gives

$$\begin{aligned}
\int_{\tilde{Q}} (u_j - u_k) \partial_t w_{jk} \tilde{\varphi} \, dx \, dt &= - \int_{\tilde{Q}} \int_{-2M}^{u_j - u_k} s \chi_{\{|s| \leq \delta\}} \, ds \, \partial_t \tilde{\varphi} \, dx \, dt \\
&\leq \int_{\tilde{Q}} \int_{-2M}^{u_j - u_k} |s| \chi_{\{|s| \leq \delta\}} \, ds |\partial_t \tilde{\varphi}| \, dx \, dt \leq 4MC|Q|\delta.
\end{aligned}$$

Since $|w_{jk}| \leq \delta$, we can estimate the second and third term by $2MC|Q|\delta$. By Hölder's inequality and Lemma 2.6 we obtain

$$\begin{aligned}
- \int_{\tilde{Q}} (\mathcal{A}(Du_j) - \mathcal{A}(Du_k)) \cdot D\tilde{\varphi} w_{jk} \, dx \, dt &\leq C\delta \int_{\tilde{Q}} |\mathcal{A}(Du_j) - \mathcal{A}(Du_k)| \, dx \, dt \\
&\leq 2C\delta \|\chi_{\tilde{Q}}\|_{L^G(\tilde{Q})} \|\mathcal{A}(Du_j) - \mathcal{A}(Du_k)\|_{L^{\tilde{G}}(\tilde{Q})} \\
&\leq 4M_2C \max\left\{|Q|^{\frac{1}{g_1}}, |Q|^{\frac{1}{g_0}}\right\} \delta.
\end{aligned}$$

Exactly the same estimate holds also for the last term in (3.4). Thus, we have

$$\int_{Q_\delta} (\mathcal{A}(Du_j) - \mathcal{A}(Du_k)) \cdot (Du_j - Du_k) \tilde{\varphi} \, dx \, dt \leq c\delta,$$

where $Q_\delta := \tilde{Q} \cap \{|u_j - u_k| \leq \delta\}$ and c depends on g_0, g_1, ν, L, M, C , and $|Q|$ but not on j and k .

Next by Lemma 2.4 and Lemma 2.2 we obtain

$$\begin{aligned}
\int_{Q_\delta} |V_g(Du_j) - V_g(Du_k)|^2 \tilde{\varphi} \, dx \, dt &\leq c \int_{Q_\delta} g'(|Du_j| + |Du_k|) |Du_j - Du_k|^2 \tilde{\varphi} \, dx \, dt \\
&\leq c \int_{Q_\delta} (\mathcal{A}(Du_j) - \mathcal{A}(Du_k)) \cdot (Du_j - Du_k) \tilde{\varphi} \, dx \, dt \leq c\delta.
\end{aligned}$$

Using Hölder's inequality and the fact that

$$\int_Q |V_g(Du_i)|^2 \, dx \, dt \leq g_1 \int_Q G(|Du_i|) \, dx \, dt \leq g_1 M_1$$

for all $i \in \mathbb{N}$ gives

$$\begin{aligned}
&\int_{Q'} |V_g(Du_j) - V_g(Du_k)| \, dx \, dt \\
&\leq \int_{Q_\delta} |V_g(Du_j) - V_g(Du_k)| \tilde{\varphi} \, dx \, dt + \int_{\tilde{Q} \setminus Q_\delta} |V_g(Du_j) - V_g(Du_k)| \tilde{\varphi} \, dx \, dt
\end{aligned}$$

$$\begin{aligned}
&\leq |Q|^{\frac{1}{2}} \left(\int_{Q_\delta} |V_g(Du_j) - V_g(Du_k)|^2 \tilde{\varphi} dx dt \right)^{\frac{1}{2}} \\
&\quad + |\tilde{Q} \setminus Q_\delta|^{\frac{1}{2}} \left(\int_Q |V_g(Du_j) - V_g(Du_k)|^2 dx dt \right)^{\frac{1}{2}} \\
&\leq c\delta^{\frac{1}{2}} + c|\tilde{Q} \setminus Q_\delta|^{\frac{1}{2}}.
\end{aligned}$$

Fix $\varepsilon > 0$ and choose δ such that $c\delta^{\frac{1}{2}} < \frac{\varepsilon}{2}$ holds. Since the sequence $(u_i)_{i=1}^\infty$ converges almost everywhere, and therefore in measure, we may choose j and k large enough such that $c|Q \setminus Q_\delta|^{\frac{1}{2}} < \frac{\varepsilon}{2}$. Thus, we have shown that the sequence $(V_g(Du_i))_{i=1}^\infty$ is Cauchy in $L^1(Q')$ and therefore there exists a function $w \in L^1(Q')$ such that $V_g(Du_i) \rightarrow w$ in $L^1(Q')$ as $i \rightarrow \infty$. This implies that there exists a subsequence $(V_g(Du_{i_j}))_{j=1}^\infty$ converging to w almost everywhere in Q' . Now the fact that V_g has a continuous inverse yields

$$Du_{i_j} = V_g^{-1}(V_g(Du_{i_j})) \rightarrow V_g^{-1}(w) =: v$$

almost everywhere in Q' .

By Fatou's lemma we obtain

$$\int_{Q'} G(|v|) dx dt \leq \liminf_{j \rightarrow \infty} \int_{Q'} G(|Du_{i_j}|) dx dt \leq M_1,$$

that is, $v \in L^G(Q')$. Now, since $u_i \rightarrow u$ almost everywhere in Q' , we have for any $\phi \in C_c^\infty(Q')$ that

$$\int_{Q'} u D\phi dx dt = \lim_{j \rightarrow \infty} \int_{Q'} u_{i_j} D\phi dx dt = - \lim_{j \rightarrow \infty} \int_{Q'} Du_{i_j} \phi dx dt = - \int_{Q'} v \phi dx dt$$

by Lebesgue's dominated convergence theorem and the definition of weak gradient, showing that $v = Du$. Thus, we have $Du_{i_j} \rightarrow Du$ almost everywhere in Q' for any $Q' \Subset Q$, which implies that $Du_{i_j} \rightarrow Du$ almost everywhere in Q .

To show that, in fact, the whole sequence $(Du_i)_{i=1}^\infty$ converges to Du almost everywhere assume the contrary. Then there exists a subsequence $(Du_{i_k})_{k=1}^\infty$ such that for some $\varepsilon' > 0$ we have $|Du_{i_k} - Du| \geq \varepsilon'$ for every k . However, the above reasoning holds if we replace u_i by u_{i_k} and thus, we find a subsubsequence $(u_{i_{k_j}})_{j=1}^\infty$ such that $|Du_{i_{k_j}} - Du| \rightarrow 0$ almost everywhere as $j \rightarrow \infty$. This is a contradiction, which proves that $Du_i \rightarrow Du$ almost everywhere in Q , and the proof is complete. \square

Theorem 3.9. *Let $Q \subset \Omega_T$, $M \geq 1$, and let $(u_i)_{i=1}^\infty$ be a sequence of weak supersolutions in Q such that $|u_i| \leq M$ for every $i \in \mathbb{N}$ and $u_i \rightarrow u$ almost everywhere in Q . Then u is a weak supersolution in Q .*

Proof. Let $\eta \in C_c^\infty(Q)$ and choose $Q' \Subset Q$ such that $\text{supp } \eta \subset Q'$. Since

$$\begin{aligned}
&\left| \int_Q \mathcal{A}(Du_i) \cdot D\eta dx dt - \int_Q \mathcal{A}(Du) \cdot D\eta dx dt \right| \\
&\leq \int_{Q'} |\mathcal{A}(Du_i) - \mathcal{A}(Du)| |D\eta| dx dt \\
&\leq \|D\eta\|_{L^\infty(Q')} \|\mathcal{A}(Du_i) - \mathcal{A}(Du)\|_{L^1(Q')}
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_Q u_i \partial_t \eta dx dt - \int_Q u \partial_t \eta dx dt \right| \leq \int_{Q'} |u_i - u| |\partial_t \eta| dx dt \\
&\leq \|\partial_t \eta\|_{L^\infty(Q')} \|u_i - u\|_{L^1(Q')},
\end{aligned}$$

it suffices to show that $u_i \rightarrow u$ and $\mathcal{A}(Du_i) \rightarrow \mathcal{A}(Du)$ in $L^1(Q')$. The former follows by Lebesgue's dominated convergence theorem, since $u_i \rightarrow u$ almost everywhere in Q and $|u_i - u| \leq 2M \in L^1(Q')$.

To show the latter, we observe that by Lemma 3.8 $Du_i \rightarrow Du$ almost everywhere in Q' . This implies that $\mathcal{A}(Du_i) \rightarrow \mathcal{A}(Du)$ almost everywhere in Q' by the continuity of \mathcal{A} . A completely analogous application of the Caccioppoli estimate as in Lemma 3.8 gives a constant $M_2 \geq 1$ independent of i such that

$$\int_{Q'} \tilde{G}(|\mathcal{A}(Du_i)|) dx dt \leq M_2.$$

Therefore, by Fatou's lemma we have

$$\int_{Q'} \tilde{G}(|\mathcal{A}(Du)|) dx dt \leq \liminf_{i \rightarrow \infty} \int_{Q'} \tilde{G}(|\mathcal{A}(Du_i)|) dx dt \leq M_2,$$

which implies

$$\|\mathcal{A}(Du)\|_{L^{\tilde{G}}(Q')}, \|\mathcal{A}(Du_i)\|_{L^{\tilde{G}}(Q')} \leq M_2.$$

Denote $E_\gamma := Q' \cap \{|\mathcal{A}(Du_i) - \mathcal{A}(Du)| \geq \gamma\}$, where $\gamma > 0$ will be chosen shortly. By Hölder's inequality and Lemma 2.6 we obtain

$$\begin{aligned} \int_{Q'} |\mathcal{A}(Du_i) - \mathcal{A}(Du)| dx dt &= \int_{Q' \setminus E_\gamma} |\mathcal{A}(Du_i) - \mathcal{A}(Du)| dx dt \\ &\quad + \int_{E_\gamma} |\mathcal{A}(Du_i) - \mathcal{A}(Du)| dx dt \\ &\leq \gamma |Q| + 2 \|\chi_{E_\gamma}\|_{L^G(Q')} \|\mathcal{A}(Du_i) - \mathcal{A}(Du)\|_{L^{\tilde{G}}(Q')} \\ &\leq \gamma |Q| + 4M_2 \max\left\{|E_\gamma|^{\frac{1}{g_1}}, |E_\gamma|^{\frac{1}{g_0}}\right\}. \end{aligned}$$

Now, for a fixed $\varepsilon > 0$, we first choose $\gamma = \frac{\varepsilon}{2|Q|}$ and then i large enough such that

$$4M_2 \max\left\{|E_\gamma|^{\frac{1}{g_1}}, |E_\gamma|^{\frac{1}{g_0}}\right\} < \frac{\varepsilon}{2}.$$

This can be done, since in a set with finite measure convergence almost everywhere implies convergence in measure. Thus, we have shown that $\mathcal{A}(Du_i) \rightarrow \mathcal{A}(Du)$ in $L^1(Q')$ and the proof is complete. \square

4. QUALITATIVE PROPERTIES OF SOLUTIONS

In this section we prove that weak supersolutions always have a lower semicontinuous representative. For this we need boundedness of nonnegative weak subsolutions which is also an interesting result in its own right. For the evolutionary p -Laplace equation the lower semicontinuity of weak supersolutions was first proved in [25]. We remark that the results of this section hold also for more general vector fields $\mathcal{A}(x, t, \xi)$ being measurable in (x, t) , continuous in ξ , and satisfying the weaker structural conditions (2.11).

In order to choose the correct geometry we need to understand the scaling of the equation. Suppose u is a weak solution in $B_\rho \times (-\theta, 0)$. Then

$$\bar{u}(x, t) := \frac{u(\rho x, \theta t)}{k}$$

is a weak solution in $B_1 \times (-1, 0)$ with \mathcal{A} replaced by

$$\bar{\mathcal{A}}(\xi) := \frac{k}{\rho} G\left(\frac{k}{\rho}\right)^{-1} \mathcal{A}\left(\frac{k}{\rho} \xi\right)$$

if and only if

$$\theta = k^2 G\left(\frac{k}{\rho}\right)^{-1}. \quad (4.1)$$

Observe that $\bar{\mathcal{A}}$ satisfies the same structural conditions as \mathcal{A} with g replaced by

$$\bar{g}(s) := \frac{k}{\rho} G\left(\frac{k}{\rho}\right)^{-1} g\left(\frac{k}{\rho} s\right),$$

and furthermore, \bar{g} satisfies the Orlicz condition (1.2) with the same constants as g .

We begin by proving an *a priori* result using a standard De Giorgi iteration.

Lemma 4.1. *Let $(x_0, t_0) \in \Omega_T$, $k > 0$, and take $\rho, \theta > 0$ such that*

$$Q(\rho, \theta) := B_\rho(x_0) \times (t_0 - \theta, t_0) \Subset \Omega_T$$

and (4.1) holds. If u is a nonnegative weak subsolution in Ω_T , then there exists a constant $\sigma \equiv \sigma(n, g_0, g_1, \nu, L) \in (0, 1)$ such that whenever

$$\int_{Q(\rho, \theta)} \left(G\left(\frac{u}{\rho}\right) + \frac{u^2}{\theta} \right) dx dt \leq \sigma G\left(\frac{k}{\rho}\right) \quad (4.2)$$

holds, we have

$$\operatorname{ess\,sup}_{Q(\rho/2, \theta/2)} u \leq k.$$

Proof. Set for $j \in \mathbb{N}_0$

$$\rho_j = (1 + 2^{-j}) \frac{\rho}{2}, \quad \theta_j = (1 + 2^{-j}) \frac{\theta}{2}, \quad k_j = (1 - 2^{-j})k$$

and define

$$Q_j := Q(\rho_j, \theta_j).$$

Moreover, for technical reasons we introduce

$$\tilde{\rho}_j = \frac{\rho_j + \rho_{j+1}}{2}, \quad \tilde{\theta}_j = \frac{\theta_j + \theta_{j+1}}{2}, \quad \tilde{Q}_j = Q(\tilde{\rho}_j, \tilde{\theta}_j).$$

Observe that $Q_{j+1} \subset \tilde{Q}_j \subset Q_j$. Let $\varphi_j \in C^\infty(Q_j)$, $\tilde{\varphi}_j \in C^\infty(\tilde{Q}_j)$ be such that $\varphi_j, \tilde{\varphi}_j$ vanish on $\partial_p Q_j, \partial_p \tilde{Q}_j$, respectively, $0 \leq \varphi_j, \tilde{\varphi}_j \leq 1$, $\varphi_j = 1$ in \tilde{Q}_j , $\tilde{\varphi}_j = 1$ in Q_{j+1} , and

$$|D\varphi_j|, |D\tilde{\varphi}_j| \leq c \frac{2^j}{\rho}, \quad |\partial_t \varphi_j|, |\partial_t \tilde{\varphi}_j| \leq c \frac{2^j}{\theta},$$

where c is a universal constant.

Define

$$Y_j := G\left(\frac{k}{\rho}\right)^{-1} \int_{Q_j} \left(G\left(\frac{(u - k_j)_+}{\rho}\right) + \frac{(u - k_j)_+^2}{\theta} \right) dx dt.$$

The aim is to use De Giorgi's iteration method and for that we need to estimate Y_{j+1} . We shall estimate the two integral terms appearing in Y_{j+1} separately. Since in the support of $(u - k_{j+1})_+$ we have

$$(u - k_j)_+ \geq k_{j+1} - k_j = 2^{-j-1}k, \quad (4.3)$$

Lemma 2.8 with $q = 1$ yields

$$\begin{aligned} & \int_{Q_{j+1}} G\left(\frac{(u - k_{j+1})_+}{\rho}\right) dx dt \\ & \leq c 2^{2j/n} k^{-2/n} \int_{\tilde{Q}_j} G\left(\frac{(u - k_j)_+ \tilde{\varphi}_j}{\tilde{\rho}_j}\right) ((u - k_j)_+ \tilde{\varphi}_j)^{2/n} dx dt \\ & \leq c 2^{2j/n} k^{-2/n} \operatorname{ess\,sup}_{(t_0 - \tilde{\theta}_j, t_0)} \left(\int_{B_{\tilde{\rho}_j}} (u - k_j)_+^2 \tilde{\varphi}_j^2 dx \right)^{1/n} \\ & \quad \times \int_{\tilde{Q}_j} G(|D((u - k_j)_+ \tilde{\varphi}_j)|) dx dt \\ & \leq c 2^{2j/n} k^{-2/n} \theta^{1/n} \left(\frac{1}{\theta_j} \operatorname{ess\,sup}_{(t_0 - \theta_j, t_0)} \int_{B_{\rho_j}} (u - k_j)_+^2 \varphi_j^{g_1} dx \right)^{1/n} \end{aligned}$$

$$\times \left(\int_{Q_j} G(|D(u - k_j)_+|) \varphi_j^{g_1} dx dt + \int_{Q_j} G((u - k_j)_+ |D\tilde{\varphi}_j|) dx dt \right).$$

The Caccioppoli inequality, Lemma 3.3, gives

$$\begin{aligned} & \frac{1}{\theta_j} \operatorname{ess\,sup}_{(t_0 - \theta_j, t_0)} \int_{B_{\rho_j}} (u - k_j)_+^2 \varphi_j^{g_1} dx + \int_{Q_j} G(|D(u - k_j)_+|) \varphi_j^{g_1} dx dt \\ & \leq c \int_{Q_j} G(|D\varphi_j|(u - k_j)_+) dx dt + c \int_{Q_j} (u - k_j)_+^2 |\partial_t \varphi_j| dx dt, \end{aligned}$$

since φ_j vanishes at $t = t_0 - \theta_j$, and thus, using also (4.1), we obtain

$$\begin{aligned} & \int_{Q_{j+1}} G\left(\frac{(u - k_{j+1})_+}{\rho}\right) dx dt \\ & \leq c 2^{2j/n} k^{-2/n} \theta^{1/n} \left(\int_{Q_j} G\left(2^j \frac{(u - k_j)_+}{\rho}\right) dx dt + \int_{Q_j} 2^j \frac{(u - k_j)_+^2}{\theta} dx dt \right)^{1+1/n} \\ & \leq c 2^{(2/n + g_1(1+1/n))j} \left[\theta k^{-2} G\left(\frac{k}{\rho}\right) \right]^{1/n} G\left(\frac{k}{\rho}\right) Y_j^{1+1/n} = c b_1^j G\left(\frac{k}{\rho}\right) Y_j^{1+1/n}, \end{aligned}$$

where $b_1 = 2^{2/n + g_1(1+1/n)}$.

For the second term we apply Lemma 2.8 with $q = 2n/(n+2)$. The mapping $s \mapsto sG(s)^{-1/q}$ is decreasing due to the assumption $g_0 > 2n/(n+2)$, and therefore by (4.3) in the support of $(u - k_{j+1})_+$ we have

$$\frac{(u - k_j)_+}{\rho} G\left(\frac{(u - k_j)_+}{\rho}\right)^{-1/q} \leq \frac{k}{2^{j+1}\rho} G\left(\frac{k}{2^{j+1}\rho}\right)^{-1/q},$$

which implies

$$(u - k_j)_+ \leq 2^{(g_1/q - 1)(j+1)} k G\left(\frac{k}{\rho}\right)^{-1/q} G\left(\frac{(u - k_j)_+}{\tilde{\rho}_j}\right)^{1/q}.$$

Combining this with the Caccioppoli inequality as above yields

$$\begin{aligned} & \int_{Q_{j+1}} \frac{(u - k_{j+1})_+^2}{\theta} dx dt \\ & \leq c 2^{(g_1/q - 1)j} \theta^{-1} k G\left(\frac{k}{\rho}\right)^{-1/q} \int_{\tilde{Q}_j} G\left(\frac{(u - k_j)_+ \tilde{\varphi}_j}{\tilde{\rho}_j}\right)^{1/q} (u - k_j)_+ \tilde{\varphi}_j dx dt \\ & \leq c 2^{(g_1/q - 1)j} \theta^{-1} k G\left(\frac{k}{\rho}\right)^{-1/q} \operatorname{ess\,sup}_{(t_0 - \tilde{\theta}_j, t_0)} \left(\int_{B_{\tilde{\rho}_j}} (u - k_j)_+^2 \tilde{\varphi}_j^2 dx \right)^{1/2} \\ & \quad \times \left(\int_{\tilde{Q}_j} G(|D((u - k_j)_+ \tilde{\varphi}_j)|) dx dt \right)^{1/q} \\ & \leq c 2^{(g_1/q - 1)j} \theta^{-1/2} k G\left(\frac{k}{\rho}\right)^{-1/q} \left(\frac{1}{\theta_j} \operatorname{ess\,sup}_{(t_0 - \theta_j, t_0)} \int_{B_{\rho_j}} (u - k_j)_+^2 \varphi_j^{g_1} dx \right)^{1/2} \\ & \quad \times \left(\int_{Q_j} G(|D(u - k_j)_+|) \varphi_j^{g_1} dx dt + \int_{Q_j} G((u - k_j)_+ |D\tilde{\varphi}_j|) dx dt \right)^{1/q} \\ & \leq c 2^{((3/2 + 2/n)g_1 - 1)j} \left[\theta k^{-2} G\left(\frac{k}{\rho}\right) \right]^{-1/2} G\left(\frac{k}{\rho}\right) Y_j^{1+1/n} = c b_2^j G\left(\frac{k}{\rho}\right) Y_j^{1+1/n}, \end{aligned}$$

where $b_2 = 2^{(3/2 + 2/n)g_1 - 1}$.

By putting the two estimates together we obtain

$$\begin{aligned} Y_{j+1} &= G\left(\frac{k}{\rho}\right)^{-1} \left(\int_{Q_{j+1}} G\left(\frac{(u - k_{j+1})_+}{\rho}\right) dx dt + \int_{Q_{j+1}} \frac{(u - k_{j+1})_+^2}{\theta} dx dt \right) \\ &\leq c b^j Y_j^{1+1/n}, \end{aligned}$$

where $b = \max\{b_1, b_2\}$. Now a standard hyper-geometric iteration lemma, see Lemma 4.1 in [13], yields $Y_j \rightarrow 0$ as $j \rightarrow \infty$ provided that $Y_0 \leq c^{-n} b^{-n^2}$. But this condition is precisely (4.2) if we choose $\sigma = c^{-n} b^{-n^2}$. Therefore $u \leq k$ almost everywhere in $Q(\rho/2, \theta/2)$, as required. \square

In order to prove the boundedness of nonnegative weak subsolutions we still need to show that there always exists a number k that satisfies (4.2). Due to the general nature of the equation this can only be done implicitly so that, at least with our method, it is not possible to obtain a nice *a priori* estimate, like in the case of the p -Laplacian (see [13]).

To this end, define

$$a := \liminf_{s \rightarrow \infty} \frac{s^2}{G(s)}, \quad A := \limsup_{s \rightarrow \infty} \frac{s^2}{G(s)}.$$

Depending on the growth of G we consider three separate cases. The case $a = A = 0$ is the *degenerate* case. When $a = A = \infty$ we have the *singular* case. The remaining case where either a or A is strictly positive and finite, or $a = 0$ and $A = \infty$, we shall call the *intermediate* case. Notice that when $g_0 > 2$ ($g_1 < 2$) we must be in the degenerate (singular) case, and on the other hand in the degenerate (singular) case we always have $g_1 > 2$ ($g_0 < 2$).

Theorem 4.2. *Let u be a nonnegative weak subsolution in Ω_T . Then $u \in L_{loc}^\infty(\Omega_T)$.*

Proof. The idea is to show that in each case there exists some finite k and a neighborhood $Q(\rho, \theta)$ of $(x_0, t_0) \in \Omega_T$ such that (4.2) holds. Then

$$\operatorname{ess\,sup}_{Q(\rho/2, \theta/2)} u \leq k$$

by Lemma 4.1 and the claim follows. Recall that the notion of weak subsolution includes that $u \in V_{loc}^{2,G}(\Omega_T)$, and hence in particular

$$\int_Q (G(u) + u^2) dx dt < \infty$$

for any $Q \Subset \Omega_T$.

Intermediate case. If either $1 \leq a < \infty$ or $0 < A \leq 1$ there clearly exist a constant $1 \leq M < \infty$ and a sequence $(s_m)_{m=0}^\infty$ such that $\lim_{m \rightarrow \infty} s_m = \infty$ and for every $m \in \mathbb{N}_0$ we have

$$\frac{1}{M} \leq \frac{s_m^2}{G(s_m)} \leq M. \quad (4.4)$$

On the other hand, if $a < 1 < A$ we can always find a sequence $(s_m)_{m=0}^\infty$ such that $\lim_{m \rightarrow \infty} s_m = \infty$ and $\frac{s_m^2}{G(s_m)} = 1$ for every $m \in \mathbb{N}_0$, and thus in this case (4.4) holds with $M = 1$.

Fix a radius $0 < r < 1/M$ such that $Q(r, r) \Subset \Omega_T$ and let m^* be the smallest m that satisfies

$$s_m \geq \left(\frac{1}{\sigma r} \int_{Q(r, r)} \left(G\left(\frac{u}{r}\right) + M\left(\frac{u}{r}\right)^2 \right) dx dt \right)^{1/2}.$$

Set $k = rs_{m^*}$ and choose $\rho = r$ and $\theta = k^2 G\left(\frac{k}{r}\right)^{-1}$. Observe that (4.1) holds and, moreover,

$$\theta = \frac{k^2}{G\left(\frac{k}{r}\right)} = \frac{s_{m^*}^2}{G(s_{m^*})} r^2 \leq Mr^2 \leq r$$

by (4.4). Now

$$\begin{aligned} \int_{Q(\rho, \theta)} \left(G\left(\frac{u}{\rho}\right) + \frac{u^2}{\theta} \right) dx dt &= \frac{G\left(\frac{k}{r}\right)}{|B_r|k^2} \int_{Q(\rho, \theta)} \left(G\left(\frac{u}{r}\right) + \frac{G(s_{m^*})}{s_{m^*}^2} \left(\frac{u}{r}\right)^2 \right) dx dt \\ &\leq \frac{G\left(\frac{k}{r}\right)}{rs_{m^*}^2} \int_{Q(r, r)} \left(G\left(\frac{u}{r}\right) + M\left(\frac{u}{r}\right)^2 \right) dx dt \leq \sigma G\left(\frac{k}{\rho}\right). \end{aligned}$$

Degenerate case. Since now $\limsup_{s \rightarrow \infty} \frac{s^2}{G(s)} = 0$, there exists $s_0 \geq 1$ such that

$$\sup_{s \geq s_0} \frac{s^2}{G(s)} \leq 1. \quad (4.5)$$

Set for $m \in \mathbb{N}_0$

$$D_m := \sup_{s \geq s_0 + m} \frac{s^2}{G(s)}$$

and observe that $D_m \leq D_0 \leq 1$ by (4.5). Using the very definition of supremum, for every $m \in \mathbb{N}_0$ there exists $s_m \geq s_0 + m$ such that

$$\frac{s_m^2}{G(s_m)} \geq \frac{1}{2} D_m. \quad (4.6)$$

Let $r > 0$ be such that $Q(r, r^2) \Subset \Omega_T$. We clearly have $\lim_{m \rightarrow \infty} s_m = \infty$ so that we may find the smallest m , which we again call m^* , such that

$$s_m \geq \left(\frac{c^*}{\sigma} \int_{Q(r, r^2)} G\left(\frac{u}{r}\right) dx dt \right)^{1/2}.$$

The constant c^* shall be determined shortly. Again, set $k = rs_{m^*}$ and choose $\rho = r$ and $\theta = k^2 G\left(\frac{k}{r}\right)^{-1}$. Since $\frac{s_{m^*}^2}{G(s_{m^*})} \leq D_m \leq 1$, we have

$$\theta = \frac{k^2}{G\left(\frac{k}{r}\right)} = \frac{s_{m^*}^2}{G(s_{m^*})} r^2 \leq r^2.$$

Take $0 < \varepsilon < 1$ to be chosen later. In the set $\{u \geq \varepsilon k\}$ we obtain by a change of variables and (4.6) that

$$\begin{aligned} \frac{u^2}{G\left(\frac{u}{r}\right)} &\leq \sup_{s \geq \varepsilon k} \frac{s^2}{G\left(\frac{s}{r}\right)} = \varepsilon^2 r^2 \sup_{s \geq s_{m^*}} \frac{s^2}{G(s)} \\ &\leq \varepsilon^{2-g_1} r^2 D_{m^*} \leq 2\varepsilon^{2-g_1} r^2 \frac{s_{m^*}^2}{G(s_{m^*})} = 2\varepsilon^{2-g_1} \theta. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{Q(\rho, \theta)} \left(G\left(\frac{u}{\rho}\right) + \frac{u^2}{\theta} \right) dx dt &= \int_{Q(\rho, \theta)} \left(G\left(\frac{u}{r}\right) + \frac{u^2}{k^2} G\left(\frac{k}{r}\right) \right) \chi_{\{u < \varepsilon k\}} dx dt \\ &\quad + \frac{G\left(\frac{k}{r}\right)}{|B_r|k^2} \int_{Q(\rho, \theta)} \left(G\left(\frac{u}{r}\right) + \frac{u^2}{\theta} \right) \chi_{\{u \geq \varepsilon k\}} dx dt \\ &\leq \left(\varepsilon^{g_0} + \varepsilon^2 + \frac{1 + 2\varepsilon^{2-g_1}}{s_{m^*}^2} \int_{Q(r, r^2)} G\left(\frac{u}{r}\right) dx dt \right) G\left(\frac{k}{r}\right) \\ &\leq \left(2\varepsilon^{\min\{g_0, 2\}} + \frac{1 + 2\varepsilon^{2-g_1}}{c^*} \sigma \right) G\left(\frac{k}{r}\right) = \sigma G\left(\frac{k}{\rho}\right), \end{aligned}$$

if we choose $\varepsilon = (\sigma/4)^{1/\min\{g_0, 2\}}$ and $c^* = 2(1 + 2\varepsilon^{2-g_1})$.

Singular case. The proof is very similar to the degenerate case and therefore some details shall be omitted. A change of variables gives

$$\liminf_{s \rightarrow \infty} \frac{s^2}{G(s)} = \left(\limsup_{s \rightarrow \infty} \frac{s}{G^{-1}(s^2)} \right)^{-2},$$

and thus the condition $a = \infty$ is equivalent to $\limsup_{s \rightarrow \infty} \frac{s}{G^{-1}(s^2)} = 0$. Proceeding as in the degenerate case, choose $s_0 \geq 1$ such that

$$\sup_{s \geq s_0} \frac{s}{G^{-1}(s^2)} \leq 1,$$

set for $m \in \mathbb{N}_0$

$$S_m := \sup_{s \geq s_0 + m} \frac{s}{G^{-1}(s^2)},$$

and construct the sequence $(s_m)_{m=0}^\infty$ such that $s_m \geq s_0 + m$ and

$$\frac{s_m}{G^{-1}(s_m^2)} \geq \frac{1}{2} S_m.$$

We again fix $r > 0$ such that $Q(r, r^2) \Subset \Omega_T$ and this time take m^* to be the smallest m for which

$$s_m \geq \left(\frac{c^*}{\sigma} \int_{Q(r, r^2)} \frac{u^2}{r^2} dx dt \right)^{1/(n+2-2n/g_0)}$$

holds for some c^* to be fixed. Set $k = r s_{m^*}$ and choose $\rho = k \left[G^{-1} \left(\frac{k^2}{r^2} \right) \right]^{-1}$ and $\theta = r^2$.

Notice that (4.1) holds and we again have $Q(\rho, \theta) \subset Q(r, r^2)$, since

$$\rho = \frac{k}{G^{-1} \left(\frac{k^2}{r^2} \right)} = \frac{s_{m^*}}{G^{-1}(s_{m^*}^2)} r \leq S_{m^*} r \leq r.$$

Let ε be the same as above. A similar calculation as before shows that

$$\frac{u}{G^{-1} \left(\frac{u^2}{r^2} \right)} \leq 2\varepsilon^{1-2/g_0} \rho$$

in the set $\{u \geq \varepsilon k\}$. Thus

$$\begin{aligned} \int_{Q(\rho, \theta)} \left(G \left(\frac{u}{\rho} \right) + \frac{u^2}{\theta} \right) dx dt &= \int_{Q(\rho, \theta)} \left(G \left(\frac{u}{k} G^{-1} \left(\frac{k^2}{r^2} \right) \right) + \frac{u^2}{r^2} \right) \chi_{\{u < \varepsilon k\}} dx dt \\ &\quad + \frac{[G^{-1}(s_{m^*}^2)]^n}{|B_r| r^2 s_{m^*}^n} \int_{Q(\rho, \theta)} \left(G \left(\frac{u}{\rho} \right) + \frac{u^2}{r^2} \right) \chi_{\{u \geq \varepsilon k\}} dx dt \\ &\leq \left(\varepsilon^{g_0} + \varepsilon^2 + s_{m^*}^{2n/g_0 - (n+2)} \left((2\varepsilon^{1-2/g_0})^{g_1} + 1 \right) \int_{Q(r, r^2)} \frac{u^2}{r^2} dx dt \right) \frac{k^2}{r^2} \\ &\leq \left(2\varepsilon^{\min\{g_0, 2\}} + \frac{(2\varepsilon^{1-2/g_0})^{g_1} + 1}{c^*} \sigma \right) \frac{k^2}{r^2} = \sigma G \left(\frac{k}{\rho} \right) \end{aligned}$$

upon choosing $c^* = 2 \left((2\varepsilon^{1-2/g_0})^{g_1} + 1 \right)$.

We have shown that in all three cases a finite k exists, which proves that nonnegative weak subsolutions are locally bounded. \square

Remark 4.3. Observe that we get no quantitative information about the size of k in any of the cases. This is due to the fact that we only have qualitative information about the sequence s_m , so that s_{m^*} could be arbitrarily large, although finite.

If u is only a weak subsolution but not necessarily nonnegative, we may apply the result to $\max\{u, 0\}$ which is a nonnegative weak subsolution by Lemma 3.7. Similarly, if u is a weak supersolution, then $\max\{-u, 0\}$ is a nonnegative weak subsolution. Hence we obtain the following corollary.

Corollary 4.4. *Let u be a weak supersolution (subsolution) in Ω_T . Then u is locally essentially bounded from below (above). In particular, if u is a weak solution in Ω_T , then $u \in L_{loc}^\infty(\Omega_T)$.*

After the boundedness of nonnegative weak subsolutions has been established, we obtain the lower semicontinuity of supersolutions fairly easily by using the *a priori* estimate in (4.2) and Lebesgue's differentiation theorem. For this we define the ess lim inf-regularization of a function u that is bounded from below as

$$\hat{u}(x, t) := \lim_{r \rightarrow 0} \operatorname{ess\,inf}_{Q_r(x, t) \cap \Omega_T} u, \quad (4.7)$$

where $Q_r(x, t) := B_r(x) \times (t - \frac{1}{2}r^2, t + \frac{1}{2}r^2)$. First we need a simple lemma.

Lemma 4.5. *Let u be essentially bounded from below. Then \hat{u} is lower semicontinuous.*

Proof. Fix $(x, t) \in \Omega_T$ and $\varepsilon > 0$. There exists $\rho_0 > 0$ such that $Q_{\rho_0}(x, t) \subset \Omega_T$ and

$$\left| \hat{u}(x, t) - \operatorname{ess\,inf}_{Q_\rho(x, t)} u \right| < \varepsilon$$

for every $0 < \rho \leq \rho_0$. Fix such a ρ and let $(y, s) \in Q_\rho(x, t)$. Observe that for all small enough $r > 0$ we have $Q_r(y, s) \subset Q_\rho(x, t)$ and thus

$$\hat{u}(y, s) = \lim_{r \rightarrow 0} \operatorname{ess\,inf}_{Q_r(y, s)} u \geq \operatorname{ess\,inf}_{Q_\rho(x, t)} u > \hat{u}(x, t) - \varepsilon.$$

Now

$$\liminf_{(y, s) \rightarrow (x, t)} \hat{u}(y, s) = \lim_{\rho \rightarrow 0} \inf_{Q_\rho(x, t)} \hat{u} \geq \hat{u}(x, t) - \varepsilon$$

and the result follows by taking $\varepsilon \rightarrow 0$. \square

Theorem 4.6. *Let u be a weak supersolution in Ω_T . Then $u = \hat{u}$ almost everywhere in Ω_T and, in particular, u is lower semicontinuous after a redefinition in a set of measure zero.*

Proof. Since u is bounded from below by Corollary 4.4, \hat{u} is lower semicontinuous by Lemma 4.5. Thus, it suffices to show that $u = \hat{u}$ almost everywhere in Ω_T .

Assume without loss of generality that $u \in L_{loc}^\infty(\Omega_T)$. Indeed, by Lemma 3.7 the function $u_m := \min\{u, m\}$ is a weak supersolution for every $m \in \mathbb{N}$ and, furthermore, $u_m \in L_{loc}^\infty(\Omega_T)$ by Corollary 4.4. Therefore, if we show that $u_m = \hat{u}_m$ almost everywhere in Ω_T , then by the inclusion

$$\begin{aligned} & \{(x, t) \in \Omega_T : u(x, t) \neq \hat{u}(x, t)\} \\ & \subset \{(x, t) \in \Omega_T : |u(x, t)| = \infty\} \cup \bigcup_{m=1}^{\infty} \{(x, t) \in \Omega_T : u_m(x, t) \neq \hat{u}_m(x, t)\} \end{aligned}$$

and the fact that as an integrable function u is finite almost everywhere in Ω_T it follows that $u = \hat{u}$ almost everywhere in Ω_T .

Set $U := \{(x, t) \in \Omega_T : |u(x, t)| < \infty\}$ and denote the set of Lebesgue points of u in Ω_T by V . Since almost every point is a Lebesgue point and u is integrable, we clearly have $|\Omega_T \setminus (U \cap V)| = 0$. Hence, by showing that $u = \hat{u}$ in $U \cap V$ we obtain the result.

To this end, fix $\varepsilon > 0$ and take $0 < k < \varepsilon$. Let $(x_0, t_0) \in U \cap V$ and denote $\tilde{Q}(\rho, \theta) := B_\rho(x_0) \times (t_0 - \frac{1}{2}\theta, t_0 + \frac{1}{2}\theta)$, where ρ and θ are chosen such that (4.2) holds. Observe that Lemma 4.1 clearly holds also for cylinders of the type $\tilde{Q}(\rho, \theta) \Subset \Omega_T$. We define the nonnegative weak subsolution

$$v := (u(x_0, t_0) - u)_+$$

and aim to show that

$$\int_{\tilde{Q}(\rho, \theta)} \left(G\left(\frac{v}{\rho}\right) + \frac{v^2}{\theta} \right) dx dt \leq \sigma G\left(\frac{k}{\rho}\right)$$

for ρ small enough. Take $\delta > 0$ to be fixed shortly and let $\mathcal{K} \Subset \Omega_T$ be a set including (x_0, t_0) . Since v is locally bounded by Theorem 4.2, we may take N to be the smallest positive integer such that $\|v\|_{L^\infty(\mathcal{K})} < 2^N \delta$. Now for every $\rho > 0$ satisfying $\tilde{Q}(\rho, \theta) \subset \mathcal{K}$ we have

$$\begin{aligned} \int_{\tilde{Q}(\rho, \theta)} G\left(\frac{v}{\rho}\right) \chi_{\{v \geq \delta\}} dx dt &= \sum_{j=0}^{N-1} \int_{\tilde{Q}(\rho, \theta)} G\left(\frac{v}{\rho}\right) \chi_{\{2^j \delta \leq v < 2^{j+1} \delta\}} dx dt \\ &\leq \sum_{j=0}^{N-1} G\left(\frac{2^{j+1} \delta}{\rho}\right) \int_{\tilde{Q}(\rho, \theta)} \chi_{\{2^j \delta \leq v < 2^{j+1} \delta\}} dx dt \\ &\leq \sum_{j=0}^{N-1} G\left(\frac{2^{j+1} \delta}{\rho}\right) \left(\int_{\tilde{Q}(\rho, \theta)} \chi_{\{2^j \delta \leq v < 2^{j+1} \delta\}} dx dt \right)^{1/g_1} \\ &\leq \sum_{j=0}^{N-1} G\left(\frac{2}{\rho}\right) \left(\int_{\tilde{Q}(\rho, \theta)} v^{g_1} dx dt \right)^{1/g_1} \\ &\leq G\left(\frac{2N}{\rho}\right) \left(\int_{\tilde{Q}(\rho, \theta)} |u(x_0, t_0) - u|^{g_1} dx dt \right)^{1/g_1}. \end{aligned}$$

Since (x_0, t_0) is a Lebesgue point and u belongs to $L^p(\mathcal{K})$ for every $1 \leq p \leq \infty$, Lebesgue's differentiation theorem gives

$$\lim_{\rho \rightarrow 0} \int_{\tilde{Q}(\rho, \theta)} |u(x_0, t_0) - u|^p dx dt = 0$$

for $1 \leq p < \infty$. In particular, we use this for $p = g_1$ and $p = 2$ to find $\rho_0 \equiv \rho_0(g_1, \sigma, k, N)$ and $\theta_0 = k^2 G\left(\frac{k}{\rho_0}\right)^{-1}$ such that $\tilde{Q}(\rho_0, \theta_0) \subset \mathcal{K}$,

$$\int_{\tilde{Q}(\rho_0, \theta_0)} |u(x_0, t_0) - u|^{g_1} dx dt \leq \left(\frac{\sigma}{8N} k\right)^{g_1},$$

and

$$\int_{\tilde{Q}(\rho_0, \theta_0)} |u(x_0, t_0) - u|^2 dx dt \leq \frac{\sigma}{2} k^2.$$

Thus, by choosing $\delta = \frac{\sigma}{4} k$ we obtain the desired inequality

$$\begin{aligned} \int_{\tilde{Q}(\rho_0, \theta_0)} \left(G\left(\frac{v}{\rho_0}\right) + \frac{v^2}{\theta_0} \right) dx dt &\leq G\left(\frac{\delta}{\rho_0}\right) + \int_{\tilde{Q}(\rho_0, \theta_0)} G\left(\frac{v}{\rho_0}\right) \chi_{\{v \geq \delta\}} dx dt \\ &\quad + \frac{1}{k^2} G\left(\frac{k}{\rho_0}\right) \int_{\tilde{Q}(\rho_0, \theta_0)} |u(x_0, t_0) - u|^2 dx dt \\ &\leq 2 G\left(\frac{\sigma}{4} \frac{k}{\rho_0}\right) + \frac{\sigma}{2} G\left(\frac{k}{\rho_0}\right) \\ &\leq \sigma G\left(\frac{k}{\rho_0}\right). \end{aligned}$$

Let $r_0 > 0$ be so small that $Q_{r_0}(x_0, t_0) \subset \tilde{Q}(\rho_0/2, \theta_0/2)$. Then for every $0 < r \leq r_0$ we have

$$u(x_0, t_0) - \operatorname{ess\,inf}_{Q_r(x_0, t_0)} u \leq \operatorname{ess\,sup}_{Q_r(x_0, t_0)} v \leq \operatorname{ess\,sup}_{\tilde{Q}(\rho_0/2, \theta_0/2)} v \leq k < \varepsilon$$

by Lemma 4.1, and therefore

$$u(x_0, t_0) < \hat{u}(x_0, t_0) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$u(x_0, t_0) \leq \hat{u}(x_0, t_0).$$

The other direction follows from Lebesgue's differentiation theorem, since

$$u(x_0, t_0) = \lim_{r \rightarrow 0} \int_{Q_r(x_0, t_0)} u \, dx \, dt \geq \lim_{r \rightarrow 0} \operatorname{ess\,inf}_{Q_r(x_0, t_0)} u = \hat{u}(x_0, t_0),$$

and we are done. \square

5. OBSTACLE PROBLEM

In this section we prove the existence of a unique solution to the bounded obstacle problem related to equation (1.1). Moreover, we show that if the obstacle is continuous, the same property is inherited by the solution.

Definition 5.1. A function u is a solution to the obstacle problem with the obstacle ψ in Ω_T , if u is the smallest $\operatorname{ess\,lim\,inf}$ -regularized (see (4.7)) weak supersolution in Ω_T that lies above ψ almost everywhere in Ω_T .

Let us first consider merely bounded obstacles. The existence of a solution to the obstacle problem follows fairly easily using results from the previous sections. The idea of the proof is the same as in [32] for the p -Laplacian.

Theorem 5.2. *Let $\psi \in L^\infty(\Omega_T)$. Then there exists a unique solution to the obstacle problem with the obstacle ψ and, moreover, it belongs to $L^\infty(\Omega_T)$.*

Proof. By Theorem 4.6 every weak supersolution v has a representative such that $v = \hat{v}$ everywhere in Ω_T . We consider the class of all such functions that lie above ψ almost everywhere and show that the $\operatorname{ess\,lim\,inf}$ -regularization of the pointwise infimum taken over this class meets the requirements of a solution.

To this end, denote the set of all weak supersolutions in Ω_T by \mathcal{S} and define

$$\mathcal{S}_\psi := \{v \in \mathcal{S} : v = \hat{v} \text{ everywhere in } \Omega_T, v \geq \psi \text{ almost everywhere in } \Omega_T\}.$$

Since \mathcal{S}_ψ includes the constant function $v \equiv M := \|\psi\|_{L^\infty(\Omega_T)}$, it is nonempty. We set for all $(x, t) \in \Omega_T$

$$w(x, t) := \inf_{v \in \mathcal{S}_\psi} v(x, t)$$

and aim to show that \hat{w} is a solution. If v is an $\operatorname{ess\,lim\,inf}$ -regularized weak supersolution with $v \geq \psi$ almost everywhere in Ω_T , then obviously $v \geq \hat{w}$ in Ω_T . Thus, to prove that \hat{w} is a solution we need to show that $\hat{w} \in \mathcal{S}_\psi$. In fact, it suffices to show that $w \in \mathcal{S}$, since then by Theorem 4.6 $w = \hat{w}$ almost everywhere in Ω_T , which implies $\hat{w} \in \mathcal{S}_\psi$. Notice also that $\hat{w} \in L^\infty(\Omega_T)$, since $w \geq \psi \geq -M$ almost everywhere and $w \leq M$ everywhere in Ω_T , implying $|\hat{w}| \leq M$ in Ω_T .

If $v_1, v_2 \in \mathcal{S}_\psi$, then by Lemma 3.7 also $\min\{v_1, v_2\} \in \mathcal{S}_\psi$. Therefore, by Choquet's topological lemma, see p. 158 in [19], there exist a function u and a decreasing sequence $(u_i)_{i=1}^\infty$ such that $u_i \in \mathcal{S}_\psi$ for every $i \in \mathbb{N}$, $u_i \rightarrow u$ everywhere as $i \rightarrow \infty$, and

$$\liminf_{(y,s) \rightarrow (x,t)} u(y, s) = \liminf_{(y,s) \rightarrow (x,t)} w(y, s)$$

for every $(x, t) \in \Omega_T$. Clearly $u \geq w$ in Ω_T . Without loss of generality we may assume that $|u_i| \leq M$ in Ω_T for every $i \in \mathbb{N}$, and thus by Lemma 3.9 u is a weak supersolution in Ω_T . But now at almost every $(x, t) \in \Omega_T$ we know that u is lower semicontinuous by Theorem 4.6, which leads to

$$w(x, t) \leq u(x, t) \leq \liminf_{(y,s) \rightarrow (x,t)} u(y, s) = \liminf_{(y,s) \rightarrow (x,t)} w(y, s) \leq w(x, t).$$

Therefore, $w = u$ almost everywhere in Ω_T , whence $w \in \mathcal{S}$ and we deduce that \hat{w} is a solution to the obstacle problem with the obstacle ψ . Uniqueness is trivial, since \hat{w} is the smallest function in \mathcal{S}_ψ . \square

For the rest of the section we shall turn our attention to continuous obstacles. Since $C^0(\overline{\Omega}_T^p) \subset L^\infty(\Omega_T)$, the existence of a unique solution is given by Theorem 5.2. Now the interesting question is if the solution is also continuous. To answer this question we construct a sequence of functions using a modification of the Schwarz alternating method and show that the limit is indeed a continuous solution to the obstacle problem. By the uniqueness we then deduce that this solution must be the same as the one given by Theorem 5.2. Moreover, we prove that whenever the solution does not coincide with the obstacle, it is in fact a weak solution. The proof follows the same guidelines as [23] for parabolic p -Laplace type equations.

Observe that when $\psi \in C^0(\overline{\Omega}_T^p)$, the solution to the obstacle problem in fact lies above ψ everywhere.

We collect here two important results that will be needed later. They are proved in [4].

Theorem 5.3. *Let $Q := B \times \Gamma \subset \Omega_T$, where B is a ball in \mathbb{R}^n , and let $\theta \in C^0(\overline{\Omega}_T^p)$. Then there exists a unique weak solution u in Q such that $u \in C^0(\overline{Q}^p)$ and $u = \theta$ on $\partial_p Q$.*

Theorem 5.4. *Let u be the weak solution in Q given by Theorem 5.3. Then there exists a constant $c \equiv c(n, g_0, g_1, \nu, L)$ such that*

$$\|Du\|_{L^\infty(Q_R)} \leq c \left(\int_{Q_{2R}} [G(|Du|) + 1] dx dt \right)^{\max\{\frac{1}{2}, \frac{2}{(n+2)g_0 - 2n}\}}$$

for every parabolic cylinder $Q_{2R} \Subset Q$.

Let us begin with the construction of a candidate for a solution.

Construction 5.5. Let \mathcal{F} be a countable and dense family of cylinders defined as

$$\mathcal{F} = \{Q^k \subset \Omega_T : Q^k = B_{r_k}(x_k) \times (\tau_k, T), r_k, \tau_k \in \mathbb{Q}, x_k \in \mathbb{Q}^n\}.$$

Construct a sequence of functions $(\varphi_k)_{k=0}^\infty$ as follows:

$$\varphi_0 = \psi, \quad \varphi_{k+1} = \max\{\varphi_k, v_k\},$$

where v_k is a weak solution in Q^k with boundary values φ_k on $\partial_p Q^k$ and $v_k = \varphi_k$ in $\Omega_T \setminus Q^k$. Denote the limit, if it exists, by

$$u^* := \lim_{k \rightarrow \infty} \varphi_k.$$

We easily deduce the following basic properties.

Proposition 5.6. (i) *We have $\varphi_k \geq \psi$ in Ω_T for every $k \in \mathbb{N}_0$.*

(ii) *The function φ_k is continuous for every $k \in \mathbb{N}_0$.*

(iii) *We have*

$$|\varphi_k| \leq \sup_{\Omega_T} |\psi|$$

in Ω_T for every $k \in \mathbb{N}_0$.

(iv) *The limit u^* always exists and $u^* \geq \psi$ in Ω_T .*

(v) *If v is an ess lim inf-regularized weak supersolution with $v \geq \psi$ almost everywhere in Ω_T , then $v \geq u^*$ in Ω_T .*

(vi) *The limit u^* is lower semicontinuous.*

Proof. (i) By definition $\varphi_0 = \psi$ and for every $k \in \mathbb{N}$ we have

$$\varphi_k = \max\{\varphi_{k-1}, v_{k-1}\} \geq \varphi_{k-1} \geq \dots \geq \varphi_0 = \psi$$

in Ω_T . Note also that the sequence $(\varphi_k)_{k=0}^\infty$ is nondecreasing.

- (ii) The function $\varphi_0 = \psi$ is continuous by assumption. Now, if φ_k is continuous for some $k \in \mathbb{N}_0$, then so is φ_{k+1} as the maximum of continuous functions, since v_k is a weak solution in Q^k and therefore continuous by Theorem 5.3.
- (iii) Clearly $\varphi_0 = \psi \leq \sup_{\Omega_T} |\psi|$. Assume then that the claim holds for some $k \in \mathbb{N}_0$. By the maximum principle, Corollary 3.5, we have

$$|v_k| \leq \sup_{Q^k} |v_k| \leq \sup_{\partial_p Q^k} |v_k| = \sup_{\partial_p Q^k} |\varphi_k| \leq \sup_{\Omega_T} |\psi|$$

in Q^k , and thus,

$$|\varphi_{k+1}| = \begin{cases} |v_k|, & v_k > \varphi_k \\ |\varphi_k|, & v_k \leq \varphi_k \end{cases} \leq \sup_{\Omega_T} |\psi|$$

in Ω_T .

- (iv) The sequence $(\varphi_k)_{k=0}^\infty$ is nondecreasing and uniformly bounded, thus the limit u^* exists. Since all the members of the sequence are above ψ by (i), also the limit has to be.
- (v) Suppose v is an ess lim inf-regularized weak supersolution with $v \geq \psi$ almost everywhere in Ω_T . We show that $v \geq \varphi_k$ everywhere in Ω_T for every $k \in \mathbb{N}_0$, which implies $v \geq u^*$ in Ω_T . Since

$$v(x, t) = \lim_{r \rightarrow 0} \text{ess inf}_{Q_r(x, t)} v \geq \lim_{r \rightarrow 0} \text{ess inf}_{Q_r(x, t)} \psi = \psi(x, t)$$

at every $(x, t) \in \Omega_T$, the assertion holds for $\varphi_0 = \psi$. If it holds for some $k \in \mathbb{N}_0$, then by the comparison principle, Lemma 3.4, $v \geq v_k$ in Q^k , since $v \geq \varphi_k = v_k$ on $\partial_p Q^k$. Therefore, we also have $v \geq \varphi_{k+1}$ in Ω_T .

- (vi) The function u^* is the limit of a nondecreasing sequence of continuous functions, hence it is lower semicontinuous. \square

So-called \mathcal{A} -superharmonic functions are often studied in the theory of elliptic equations. As shown in [19], there is a strong connection between \mathcal{A} -superharmonic functions and weak supersolutions. The same idea can be used also in the parabolic setting. We shall call the corresponding functions \mathcal{A} -superparabolic.

Definition 5.7. A function $u : \Omega_T \rightarrow (-\infty, \infty]$ is called \mathcal{A} -superparabolic, if

- (i) u is lower semicontinuous,
- (ii) u is finite in a dense subset of Ω_T ,
- (iii) u satisfies the comparison principle in every cylinder $Q \Subset \Omega_T$, that is, if $h \in C^0(\overline{Q}^p)$ is a weak solution in Q and $h \leq u$ on $\partial_p Q$, then $h \leq u$ in Q .

In order to prove that the limit u^* of Construction 5.5 is a solution to the obstacle problem, by Proposition 5.6 it suffices to show that it is a weak supersolution. For this we prove that it is both \mathcal{A} -superparabolic and continuous. Let us first show the former.

Lemma 5.8. *The limit u^* of Construction 5.5 is \mathcal{A} -superparabolic.*

Proof. By Proposition 5.6 u^* is lower semicontinuous and

$$|u^*| = \lim_{k \rightarrow \infty} |\varphi_k| \leq \sup_{\Omega_T} |\psi|$$

in Ω_T . Thus, it is enough to show that u^* satisfies the comparison principle in every cylinder $Q \Subset \Omega_T$.

To this end, fix a cylinder $Q = K \times (t_1, t_2) \Subset \Omega_T$ and let $h \in C^0(\overline{Q}^p)$ be a weak solution in Q such that $h \leq u^*$ on $\partial_p Q$. If we can show that $h \leq u^*$ in Q , we are done. Fix $\varepsilon > 0$ and set for each $k \in \mathbb{N}$

$$E_k := \overline{Q}^p \cap \{\varphi_k > h - \varepsilon\}.$$

By the continuity of φ_k and h the sets E_k are open with respect to the relative topology. Since $u^* = \lim_{k \rightarrow \infty} \varphi_k$, for any point $z = (x, t) \in \partial_p Q$ we find an integer $k_z \geq 1$ such that

$$\varphi_{k_z}(z) > u^*(z) - \varepsilon \geq h(z) - \varepsilon,$$

implying that $z \in E_{k_z}$. Therefore, the sets E_k form an open cover for $\partial_p Q$, and since $\partial_p Q$ is compact and the sequence $(\varphi_k)_{k=0}^\infty$ nondecreasing, there exists an integer $k_0 \geq 1$ such that $\partial_p Q \subset E_{k_0}$. This, together with the fact that the set E_{k_0} is open, implies that there exists $k_1 \geq k_0$ such that the cylinder $Q^{k_1} \in \mathcal{F}$ satisfies

$$\partial_p Q^{k_1} \cap \{t < t_2\} \subset E_{k_0} \quad \text{and} \quad Q \setminus E_{k_0} \subset Q^{k_1} \cap \{t < t_2\}.$$

Now, since $v_{k_1} = \varphi_{k_1}$ on $\partial_p Q^{k_1}$, we have

$$h \leq \varphi_{k_0} + \varepsilon \leq \varphi_{k_1} + \varepsilon = v_{k_1} + \varepsilon$$

on $\partial_p Q^{k_1} \cap \{t < t_2\}$. Moreover, both h and $v_{k_1} + \varepsilon$ are weak solutions in $Q^{k_1} \cap \{t < t_2\}$, and therefore,

$$h \leq v_{k_1} + \varepsilon \leq \varphi_{k_1+1} + \varepsilon \leq u^* + \varepsilon$$

in $Q^{k_1} \cap \{t < t_2\}$ by the comparison principle, Lemma 3.4. Hence, $h \leq u^* + \varepsilon$ also in $Q \setminus E_{k_0}$, and since

$$h \leq \varphi_{k_0} + \varepsilon \leq u^* + \varepsilon$$

trivially in E_{k_0} , we obtain the result by letting ε tend to zero. \square

The next lemma shows that Construction 5.5 is stable.

Lemma 5.9. *The limit u^* of Construction 5.5 is independent of the order of the cylinders Q^k .*

Proof. Construct functions $\tilde{\varphi}_k$ and \tilde{v}_k as in Construction 5.5 with the cylinders Q^k taken in a different order than in the construction of u^* . Denote $\tilde{u}^* := \lim_{k \rightarrow \infty} \tilde{\varphi}_k$. Clearly we have $\tilde{u}^* \geq \varphi_0 = \psi$ in Ω_T . Assume then that $\tilde{u}^* \geq \varphi_k$ in Ω_T for some $k \in \mathbb{N}_0$. Since v_k is a weak solution in Q^k with $v_k = \varphi_k$ on $\partial_p Q^k$ and \tilde{u}^* is \mathcal{A} -superparabolic by Lemma 5.8, we have $v_k \leq \tilde{u}^*$ in Q^k . Thus, $\varphi_{k+1} = \max\{\varphi_k, v_k\} \leq \tilde{u}^*$ in Ω_T , and by induction we obtain $u^* \leq \tilde{u}^*$ in Ω_T . Interchanging the roles of u^* and \tilde{u}^* completes the proof. \square

This leads to the following comparison result.

Lemma 5.10. *Let ψ_1 and ψ_2 be continuous obstacles such that $\psi_1 \leq \psi_2$ in Ω_T . Then the corresponding limits u_1^* and u_2^* of Construction 5.5 satisfy $u_1^* \leq u_2^*$ in Ω_T .*

Proof. By Lemma 5.9 we may take the cylinders Q^k in the same order in the construction of both u_1^* and u_2^* . Let φ_k^i and v_k^i , $i \in \{1, 2\}$, $k \in \mathbb{N}_0$, generate u_1^* and u_2^* . By assumption $\varphi_0^1 \leq \varphi_0^2$ in Ω_T . Suppose then that $\varphi_k^1 \leq \varphi_k^2$ in Ω_T for some $k \in \mathbb{N}_0$. Then we have

$$v_k^1 = \varphi_k^1 \leq \varphi_k^2 = v_k^2$$

on $\partial_p Q^k$, and thus the comparison principle, Lemma 3.4, yields $v_k^1 \leq v_k^2$ in Q^k . This implies $\varphi_{k+1}^1 \leq \varphi_{k+1}^2$ in Ω_T , and hence, $\varphi_k^1 \leq \varphi_k^2$ in Ω_T for every $k \in \mathbb{N}_0$. Taking the limit $k \rightarrow \infty$ on both sides completes the proof. \square

We now have the necessary tools to prove the continuity of our candidate.

Lemma 5.11. *The limit u^* of Construction 5.5 is continuous.*

Proof. Let $\varepsilon > 0$ and $(x_0, t_0) \in \Omega_T$ be fixed. Since Ω_T is open and the obstacle ψ continuous, there exists $r > 0$ such that $Q_r := B_r(x_0) \times (t_0 - \frac{1}{2}r^2, t_0 + \frac{1}{2}r^2) \Subset \Omega_T$ and

$$\frac{\text{osc } \psi}{Q_r^p} \leq \frac{\varepsilon}{4}.$$

Construct the modified obstacle

$$\tilde{\psi} := \begin{cases} h & \text{in } Q_r \\ \psi & \text{in } \Omega_T \setminus Q_r \end{cases},$$

where h is a weak solution in Q_r with $h = \psi$ on $\partial_p Q_r$. By Theorem 5.3 $h \in C^0(\overline{Q_r^p})$, and thus, also $\tilde{\psi}$ is continuous. Moreover, by the maximum principle, Corollary 3.5, we have

$$h - \psi \leq \sup_{\overline{Q_r^p}} h - \psi \leq \sup_{\partial_p Q_r} \psi - \inf_{\overline{Q_r^p}} \psi \leq \frac{\text{osc } \psi}{\overline{Q_r^p}}$$

in $\overline{Q_r^p}$. Similarly, $\psi - h \leq \text{osc}_{\overline{Q_r^p}} \psi$ in $\overline{Q_r^p}$, and thus, we obtain

$$|\psi - \tilde{\psi}| \leq |\psi - h| \leq \frac{\text{osc } \psi}{\overline{Q_r^p}} \leq \frac{\varepsilon}{4}$$

in Ω_T .

Let \tilde{u}^* be the limit of Construction 5.5 with the obstacle $\tilde{\psi}$ generated by the functions $\tilde{\varphi}_k$ and \tilde{v}_k , $k \in \mathbb{N}_0$. Evidently adding a constant to the obstacle changes the corresponding limit by the same constant. Thus, since we have $\psi \leq \tilde{\psi} + \frac{\varepsilon}{4}$ and $\psi \geq \tilde{\psi} - \frac{\varepsilon}{4}$ in Ω_T , an application of Lemma 5.10 yields

$$|u^* - \tilde{u}^*| \leq \frac{\varepsilon}{4}$$

in Ω_T .

Next we prove that the function \tilde{u}^* is continuous in $Q_{r/4}$. We start by showing that the function $\tilde{\varphi}_k$ is a weak subsolution in Q_r for every $k \in \mathbb{N}_0$. This is clearly true for $k = 0$, since $\tilde{\varphi}_0 = \tilde{\psi}$ is a weak solution in Q_r . Assume then that the claim holds for some $k \in \mathbb{N}_0$. Now, the function $\max\{\tilde{\varphi}_k, \tilde{v}_k\}$ is a weak subsolution in $Q_r \cap Q^k$ by Lemma 3.7. Since trivially $\max\{\tilde{\varphi}_k, \tilde{v}_k\} \geq \tilde{\varphi}_k$, we deduce that

$$\tilde{\varphi}_{k+1} = \begin{cases} \max\{\tilde{\varphi}_k, \tilde{v}_k\} & \text{in } Q_r \cap Q^k \\ \tilde{\varphi}_k & \text{in } Q_r \setminus Q^k \end{cases}$$

is a weak subsolution in Q_r by Lemma 3.6.

Extract from the sequence $(\tilde{\varphi}_k)_{k=0}^\infty$ a subsequence $(\tilde{\varphi}_{k_i})_{i=0}^\infty$ such that $k_0 \geq 1$ and

$$\partial_p \left(Q^{k_i-1} \cap \left\{ t < t_0 + \frac{1}{2}r^2 \right\} \right) \subset Q_r \setminus Q_{r/2}$$

for every $i \in \mathbb{N}_0$. This can be done, since the collection \mathcal{F} is dense. Next, notice that $\tilde{\varphi}_{k_i-1}$ is a weak subsolution in $Q^{k_i-1} \cap \{t < t_0 + \frac{1}{2}r^2\}$ for every $i \in \mathbb{N}_0$, and thus, the comparison principle, Lemma 3.4, yields $\tilde{v}_{k_i-1} \geq \tilde{\varphi}_{k_i-1}$ in $Q^{k_i-1} \cap \{t < t_0 + \frac{1}{2}r^2\}$. Therefore, $\tilde{\varphi}_{k_i} = \tilde{v}_{k_i-1}$ in $Q^{k_i-1} \cap \{t < t_0 + \frac{1}{2}r^2\}$, and since $Q_{r/2} \subset Q^{k_i-1} \cap \{t < t_0 + \frac{1}{2}r^2\}$ for every $i \in \mathbb{N}_0$, we see that $\tilde{\varphi}_{k_i}$ is a weak solution in $Q_{r/2}$ for every $i \in \mathbb{N}_0$.

Since $\tilde{\varphi}_{k_i}$ is continuous in Ω_T , we may apply Theorem 5.4 to obtain

$$\|D\tilde{\varphi}_{k_i}\|_{L^\infty(Q_{r/8})} \leq c \left(\int_{Q_{r/4}} [G(|D\tilde{\varphi}_{k_i}|) + 1] dx dt \right)^{\max\{\frac{1}{2}, \frac{2}{(n+2)g_0-2n}\}}.$$

Moreover, since $|\tilde{\varphi}_{k_i}| \leq \sup_{\Omega_T} |\psi| =: M$ in Ω_T by the maximum principle and Proposition 5.6, the Caccioppoli inequality, Lemma 3.3, yields

$$\int_{Q_{r/4}} G(|D\tilde{\varphi}_{k_i}|) dx dt \leq c \int_{Q_{r/2}} \left[G\left(\frac{|\tilde{\varphi}_{k_i}|}{r}\right) + \frac{|\tilde{\varphi}_{k_i}|^2}{r^2} \right] dx dt \leq c(g_0, g_1, \nu, L, M, r)$$

for every $i \in \mathbb{N}_0$. Thus, we have a uniform bound for $\|D\tilde{\varphi}_{k_i}\|_{L^\infty(Q_{r/8})}$ and since $\tilde{\varphi}_{k_i}$ converges pointwise to \tilde{u}^* , by Lemma 3.8 we obtain

$$\|D\tilde{u}^*\|_{L^\infty(Q_{r/8})} \leq c(n, g_0, g_1, \nu, L, M, r).$$

By applying Theorem 3.9 to both $(\tilde{\varphi}_{k_i})_{i=0}^\infty$ and $(-\tilde{\varphi}_{k_i})_{i=0}^\infty$ we see that \tilde{u}^* is also a weak solution in $Q_{r/2}$. Now the continuity of \tilde{u}^* in a neighborhood of (x_0, t_0) follows using a completely analogous proof to Proposition 4.2 in [4] together with a simple approximation argument.

Finally, it is easy to see that

$$\sup u^* - \sup \tilde{u}^* \leq \sup |u^* - \tilde{u}^*|$$

and thus we have

$$\text{osc } u^* - \text{osc } \tilde{u}^* = \sup u^* - \sup \tilde{u}^* + \sup(-u^*) - \sup(-\tilde{u}^*) \leq 2 \sup |u^* - \tilde{u}^*|.$$

Since \tilde{u}^* is continuous in a neighborhood of (x_0, t_0) , we find $0 < \delta < r/8$ such that

$$\text{osc}_{Q_\delta} \tilde{u}^* < \frac{\varepsilon}{2}.$$

Therefore, by writing

$$\text{osc}_{Q_\delta} u^* \leq \text{osc}_{Q_\delta} \tilde{u}^* + 2 \sup_{Q_\delta} |u^* - \tilde{u}^*| < \varepsilon,$$

we see that u^* is continuous at (x_0, t_0) , as required. \square

Let us then conclude by showing that our candidate u^* is a weak supersolution. We shall do this by constructing a sequence of weak supersolutions that converge to u^* almost everywhere and then using Theorem 3.9. Observe that u^* being the limit of Construction 5.5 plays no special role in the proof, in fact, the result holds for any continuous \mathcal{A} -superparabolic function.

Let $K_0 \Subset \Omega_T$ be a cube, and let $\{K_k^j\}_{j=1}^{2^{nk}}$ denote the k^{th} generation of dyadic subcubes of K_0 . Set $Q_0 := K_0 \times (0, T)$ and $Q_k^j := K_k^j \times (0, T)$, and define for each $k \in \mathbb{N}$ the function $u_k : Q_0 \rightarrow \mathbb{R}$ such that u_k is a weak solution in Q_k^j and $u_k = u^*$ on $\partial_p Q_k^j$ for every $j \in \{1, \dots, 2^{nk}\}$.

Lemma 5.12. *The function u_k is a weak supersolution in Q_0 for every $k \in \mathbb{N}$.*

Proof. Let us first show that u_k is \mathcal{A} -superparabolic in Q_0 . Clearly u_k is continuous in $\overline{Q_0^p}$, since u^* is continuous by Lemma 5.11 and u_k is continuous up to the boundary in each Q_k^j , $j \in \{1, \dots, 2^{nk}\}$, by Theorem 5.3. Moreover, as a continuous function u_k is bounded in the compact set $\overline{Q_0^p}$. Thus, we only need to check that u_k satisfies the comparison principle in each cylinder.

Fix a cylinder $Q \Subset Q_0$ and let $h \in C^0(\overline{Q^p})$ be a weak solution in Q such that $h \leq u_k$ on $\partial_p Q$. Since u^* is \mathcal{A} -superparabolic by Lemma 5.8 and u_k is a weak solution in Q_k^j with $u_k = u^*$ on $\partial_p Q_k^j$ for every $j \in \{1, \dots, 2^{nk}\}$, we have $u_k \leq u^*$ in $\overline{Q_0^p}$. Now $h \leq u_k \leq u^*$ on $\partial_p Q$, and therefore another application of the comparison principle yields $h \leq u^*$ in Q . This, together with the fact that $u_k = u^*$ on $\partial_p Q_k^j$ for a fixed $j \in \{1, \dots, 2^{nk}\}$, implies that $h \leq u_k$ on $Q \cap \partial_p Q_k^j$. Thus, $h \leq u_k$ also on $\partial_p(Q \cap Q_k^j)$, and since h and u_k are both weak solutions in $Q \cap Q_k^j$, we obtain $h \leq u_k$ in $Q \cap Q_k^j$ by Lemma 3.4. Since this holds for every $j \in \{1, \dots, 2^{nk}\}$, we have $h \leq u_k$ in the whole Q .

Next we prove that u_k is a weak supersolution in Q_0 . Let $\tilde{Q} \subset Q_0$ be a cylinder, and fix $r \in \mathbb{R}$ and $i \in \{1, \dots, n\}$. We show that if u_k is a weak supersolution in both $Q_1 := \{(x, t) \in \tilde{Q} : x_i < r\}$ and $Q_2 := \{(x, t) \in \tilde{Q} : x_i > r\}$, then it is a weak supersolution in \tilde{Q} . Using this repeatedly will then give the desired result.

Define for $m \in \mathbb{N}$ the set $U_m := \{(x, t) \in \tilde{Q} : r - \frac{1}{m} < x_i < r + \frac{1}{m}\}$ and the function

$$v_m = \begin{cases} h_m & \text{in } U_m \\ u_k & \text{in } \tilde{Q} \setminus U_m \end{cases},$$

where h_m is a weak solution in U_m with $h_m = u_k$ on $\partial_p U_m$. Since u_k is \mathcal{A} -superparabolic, the comparison principle yields $h_m \leq u_k$ in U_m . This, together with the fact that $u_k \in C^0(\overline{Q}_1^p)$ and $h_m \in C^0(\overline{U}_m^p)$ are weak supersolutions in Q_1 and U_m , respectively, allows us to apply Lemma 3.6 to deduce that v_m is a weak supersolution in Q_1 . Similar reasoning shows that v_m is a weak supersolution also in Q_2 . Since v_m is a weak (super)solution in U_m and being a weak supersolution is a local property, v_m is a weak supersolution in the whole \tilde{Q} .

Let $M := \sup_{\tilde{Q}} |u_k|$. By the maximum principle, Corollary 3.5,

$$|h_m| \leq \sup_{\partial_p U_m} |h_m| = \sup_{\partial_p U_m} |u_k| \leq M$$

in U_m , and thus $|v_m| \leq M$ in \tilde{Q} . Moreover, by the continuity of u_k , v_m clearly converges to u_k pointwise in \tilde{Q} as $m \rightarrow \infty$. Therefore, by Theorem 3.9 u_k is a weak supersolution in \tilde{Q} . This concludes the proof. \square

Lemma 5.13. *The limit u^* of Construction 5.5 is a weak supersolution in Ω_T .*

Proof. Let (x, t) be any point in Ω_T . Since Ω_T is open, we can always find a cube $K_0 \Subset \Omega$ and $0 < t_1 < t_2 < T$ such that $(x, t) \in \frac{1}{2}K_0 \times (t_1, t_2) =: Q$, where $\frac{1}{2}K_0$ denotes the cube with the same center and half the side length as K_0 . We may assume $\text{diam}(K_0) \leq 1$ without loss of generality. By Lemma 5.12 u_k is a weak supersolution in $Q_0 = K_0 \times (0, T)$ for every $k \in \mathbb{N}$, and since

$$|u_k| \leq \sup_{Q_k^j} |u_k| \leq \sup_{\partial_p Q_k^j} |u^*| \leq \sup_{\Omega_T} |\psi| =: M$$

in Q_k^j for every $j \in \{1, \dots, 2^{n_k}\}$ by the maximum principle, the sequence $(u_k)_{k=1}^\infty$ is uniformly bounded in Q_0 . Therefore, if we manage to show that $(u_k)_{k=1}^\infty$ has a subsequence that converges to u^* almost everywhere in Q , the result follows by Theorem 3.9.

To this end, let $\varepsilon > 0$. Since u^* is continuous by Lemma 5.11, there exists $\eta \in C^\infty(\Omega_T)$ such that $|\eta - u^*| < \varepsilon$ in \overline{Q}^p . By requiring $k \geq 2$ we may label the cubes K_k^j such that

$$\bigcup_{j=1}^{2^{n(k-1)}} K_k^j = \frac{1}{2}K_0.$$

Set

$$w_k := (\eta - u_k - \varepsilon)_+$$

and observe that $w_k(\cdot, t) \in W_0^{1,G}(K_k^j)$ for almost every $t \in (t_1, t_2)$ and every $j \in \{1, \dots, 2^{n(k-1)}\}$. Thus, by applying the Poincaré's inequality, Lemma 2.7, to the function $\text{diam}(K_k^j)w_k$, we obtain

$$\begin{aligned} \int_{t_1}^{t_2} \int_{K_k^j} G(w_k) dx dt &\leq \int_{t_1}^{t_2} \int_{K_k^j} G(\text{diam}(K_k^j)|Dw_k|) dx dt \\ &\leq 2^{-g_0 k} \int_{t_1}^{t_2} \int_{K_k^j} G(|Dw_k|) dx dt, \end{aligned}$$

since we assumed $\text{diam}(K_0) \leq 1$. Therefore,

$$\begin{aligned} \int_Q G(w_k) dx dt &= \sum_{j=1}^{2^{n(k-1)}} \int_{t_1}^{t_2} \int_{K_k^j} G(w_k) dx dt \\ &\leq 2^{-g_0 k} \sum_{j=1}^{2^{n(k-1)}} \int_{t_1}^{t_2} \int_{K_k^j} G(|Dw_k|) dx dt \\ &= 2^{-g_0 k} \int_Q G(|Dw_k|) dx dt. \end{aligned} \tag{5.1}$$

Let $\varphi \in C_c^\infty(Q_0)$ be such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in Q , and

$$\|\partial_t \varphi\|_{L^\infty(Q_0)}, \|D\varphi\|_{L^\infty(Q_0)} \leq C$$

for some $C \geq 1$. By applying the Caccioppoli estimate, Lemma 3.3, to the nonnegative weak subsolution $M - u_k$ we conclude, similarly as in the proof of Lemma 3.8, that

$$\int_Q G(|Du_k|) dx dt \leq c(M^2 C + M^{g_1} C^{g_1}) |Q_0|.$$

Thus, combining this with (5.1) yields

$$\begin{aligned} |Q \cap \{w_k > \delta\}| &= \int_{Q \cap \{w_k > \delta\}} G(\delta^{-1} \delta) dx dt \\ &\leq \delta^{-g_1} \int_Q G(w_k) dx dt \leq 2^{-g_0 k} \delta^{-g_1} \int_Q G(|Dw_k|) dx dt \\ &\leq 2^{-g_0 k} \left(\frac{2}{\delta}\right)^{g_1} \left(\int_Q G(|D\eta|) dx dt + \int_Q G(|Du_k|) dx dt \right) \\ &\leq 2^{-g_0 k} c(\delta, g_0, g_1, \nu, L, M, C, |Q_0|, \|D\eta\|_{L^\infty(Q_0)}) \end{aligned}$$

for every $\delta \in (0, 1)$, implying that $w_k \rightarrow 0$ in measure in Q as $k \rightarrow \infty$. Therefore, we find a subsequence $(w_{k_i})_{i=1}^\infty$ converging to zero almost everywhere in Q as $i \rightarrow \infty$. Now, together with the choice of η , this gives

$$\lim_{i \rightarrow \infty} u_{k_i} \geq \eta - \varepsilon > u^* - 2\varepsilon$$

almost everywhere in Q . Since $u_k \leq u^*$ in Q_0 for every $k \in \mathbb{N}$, we obtain $\lim_{i \rightarrow \infty} u_{k_i} = u^*$ almost everywhere in Q by letting $\varepsilon \rightarrow 0$. This finishes the proof. \square

Theorem 5.14. *Let ψ be a continuous function in $\overline{\Omega}_T^p$. Then the solution to the obstacle problem with the obstacle ψ in Ω_T is continuous.*

Proof. By Lemma 5.13 the limit u^* of Construction 5.5 is a weak supersolution in Ω_T . Moreover, by Proposition 5.6 $u^* \geq \psi$ in Ω_T , and if also v is an ess lim inf-regularized weak supersolution with $v \geq \psi$ almost everywhere in Ω_T , then $v \geq u^*$ in Ω_T . Thus, u^* must coincide with the unique solution to the obstacle problem given by Theorem 5.2. Continuity follows from Lemma 5.11. \square

We end the section by showing that whenever the solution to the obstacle problem lies strictly above the obstacle, it is in fact a weak solution. We shall prove this for the limit u^* of Construction 5.5, which is a solution as seen above. First we need the following lemma.

Lemma 5.15. *The function φ_k is a weak subsolution in the set $\{\varphi_k > \psi\}$ for every $k \in \mathbb{N}$.*

Proof. If $\varphi_k = \psi$ for some k , the set $\{\varphi_k > \psi\}$ is empty and there is nothing to prove. Suppose thus $\varphi_k \neq \psi$ for every $k \in \mathbb{N}$. In the set $\{\varphi_1 > \psi\}$ we must have $\varphi_1 = v_0$ and hence the claim holds for $k = 1$. Assume then that φ_k is a weak subsolution in the set $\{\varphi_k > \psi\}$ for some $k \in \mathbb{N}$. First take a point $z_0 = (x_0, t_0) \in \{\varphi_k > \psi\}$. The set $\{\varphi_k > \psi\}$ is open by the continuity of φ_k and ψ and thus, we find a cylinder $Q \subseteq \{\varphi_k > \psi\} \subset \{\varphi_{k+1} > \psi\}$ such that $z_0 \in Q$. Now, the function $\max\{\varphi_k, v_k\}$ is a weak subsolution in $Q \cap Q^k$ by Lemma 3.7, and thus,

$$\varphi_{k+1} = \begin{cases} \max\{\varphi_k, v_k\} & \text{in } Q \cap Q^k \\ \varphi_k & \text{in } Q \setminus Q^k \end{cases}$$

is a weak subsolution in Q by Lemma 3.6.

Let then $z_0 \in \{\varphi_{k+1} > \psi\} \setminus \{\varphi_k > \psi\}$. Since we always have $\varphi_k \geq \psi$ and $z_0 \notin \{\varphi_k > \psi\}$, we deduce that $\varphi_k(z_0) = \psi(z_0)$. Thus, $\varphi_{k+1}(z_0) = v_k(z_0)$ or otherwise

$\varphi_{k+1}(z_0) > \psi(z_0)$ would fail to hold. Next, denote $w_k := v_k - \varphi_k$ and choose a small enough $\varepsilon > 0$ such that $w_k(z_0) > \varepsilon$. This can be done, since

$$w_k(z_0) = v_k(z_0) - \varphi_k(z_0) = \varphi_{k+1}(z_0) - \psi(z_0) > 0.$$

By the continuity of v_k and φ_k there exists $r > 0$ such that $|w_k(z) - w_k(z_0)| < \varepsilon$ for every $z \in Q_r := K_r(x_0) \times (t_0 - r, t_0 + r)$. Since $\{\varphi_{k+1} > \psi\}$ is open, we may also assume that $Q_r \subseteq \{\varphi_{k+1} > \psi\}$. Combining these facts yields $w_k(z) > w_k(z_0) - \varepsilon > 0$ for every $z \in Q_r$, showing that $v_k > \varphi_k$ in Q_r . Therefore, $\varphi_{k+1} = v_k$ is a weak solution in Q_r .

We have now found a neighborhood in which φ_{k+1} is a weak subsolution for every point of the set $\{\varphi_{k+1} > \psi\}$. The induction argument finishes the proof. \square

Proposition 5.16. *The limit u^* of Construction 5.5 is a weak solution in the set $\{u^* > \psi\}$.*

Proof. If $u^* = \psi$ there is nothing to prove. Thus, suppose $u^* \neq \psi$ and let $z_0 = (x_0, t_0) \in \Omega_T$ be such that $u^*(z_0) > \psi(z_0)$. By the continuity of u^* and ψ the set $\{u^* > \psi\}$ is open, and thus, we find a cylinder $Q \subseteq \{u^* > \psi\}$ such that $z_0 \in Q$. Since the sequence $(\varphi_k)_{k=0}^\infty$ is nondecreasing, we clearly have for every $k \in \mathbb{N}_0$ that

$$\{\varphi_k > \psi\} \subset \{\varphi_{k+1} > \psi\} \subset \{u^* > \psi\}.$$

Moreover, if $z \notin \bigcup_{k=0}^\infty \{\varphi_k > \psi\}$, we must have $\varphi_k(z) = \psi(z)$ for every $k \in \mathbb{N}_0$, and thus $z \notin \{u^* > \psi\}$. Therefore, we see that

$$\{u^* > \psi\} = \bigcup_{k=0}^\infty \{\varphi_k > \psi\}.$$

In particular, the sets $\{\varphi_k > \psi\}$ form an open cover for $\overline{Q^p}$, and since $\overline{Q^p}$ is compact, there exists an integer $k_0 \geq 1$ such that $Q \subset \{\varphi_{k_0} > \psi\}$. Now Lemma 5.15 implies that φ_k is a weak subsolution in Q for every $k \geq k_0$, and since the sequence $(\varphi_k)_{k=0}^\infty$ is uniformly bounded, we obtain by applying Lemma 3.9 to the sequence $(-\varphi_k)_{k=k_0}^\infty$ that u^* is a weak subsolution in Q . Therefore, u^* is a weak subsolution in $\{u^* > \psi\}$, and since u^* is a weak supersolution in Ω_T by Lemma 5.13, the result follows. \square

Acknowledgements. This research has been supported by the Vilho, Yrjö and Kalle Väisälä Foundation.

REFERENCES

- [1] R. A. ADAMS: *Sobolev Spaces*, Academic Press, New York, 1975.
- [2] I. ATHANASOPOULOS, L. A. CAFFARELLI, S. SALS: Regularity of the free boundary in parabolic phase-transition problems, *Acta Math.* **176** (1996), pp. 245–282.
- [3] P. BARONI: Riesz potential estimates for a general class of quasilinear equations, *Calc. Var. Partial Differential Equations* **53** (3-4) (2015), pp. 803–846.
- [4] P. BARONI, C. LINDFORS: The Cauchy-Dirichlet problem for a general class of parabolic equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, doi: 10.1016/j.anihpc.2016.03.003
- [5] L. BERS: *Mathematical aspects of subsonic and transonic gas dynamics*, Surveys in Applied Mathematics, 3, John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London 1958.
- [6] L. BOCCARDO, A. DALL’AGLIO, T. GALLOUËT, L. ORSINA: Nonlinear parabolic equations with measure data, *J. Funct. Anal.* **147** (1997), pp. 237–258.
- [7] J. BURCZAK, P. KAPLICKÝ: Interior regularity of space derivatives to an evolutionary, symmetric φ -Laplacian, *preprint*, <http://arxiv.org/abs/1507.05843v1>
- [8] L. A. CAFFARELLI: The obstacle problem revisited, *J. Fourier Anal. Appl.* **4** (1998), pp. 383–402.
- [9] L. A. CAFFARELLI, L. SILVESTRE: Regularity Theory for Fully Nonlinear Integro-Differential Equations, *Comm. Pure Appl. Math.* **62**(5) (2009), pp. 597–638.
- [10] A. CIANCHI: Boundedness of solutions to variational problems under general growth conditions, *Comm. Partial Differential Equations* **22** (1997), no. 9-10, pp. 1629–1646.
- [11] A. CIANCHI, N. FUSCO: Gradient regularity for minimizers under general growth conditions, *J. Reine Angew. Math. (Crelle’s J.)* **507** (1999), pp. 15–36.
- [12] A. CIANCHI, V. MAZ’YA: Gradient regularity via rearrangements for p -Laplacian type elliptic boundary value problems. *J. Eur. Math. Soc. (JEMS)* **16** (2014), no. 3, pp. 571–595.

- [13] E. DiBENEDETTO: *Degenerate parabolic equations*, Universitext, Springer, New York, 1993.
- [14] L. DIENING, B. STROFFOLINI, A. VERDE: Everywhere regularity of functionals with φ -growth, *Manuscripta Math.* **129** (2009), no. 4, pp. 449–481.
- [15] L. DIENING, F. ETTWEIN: Fractional estimates for non-differentiable elliptic systems with general growth, *Forum Mathematicum* (2008), no.3, pp.523–556.
- [16] G. C. DONG: *Nonlinear partial differential equations of second order*, Translations of Mathematical Monographs, 95, American Mathematical Society, Providence, RI, 1991.
- [17] R. FINN, D. GILBARG: Three-dimensional subsonic flows, and asymptotic estimates for elliptic partial differential equations, *Acta Math.* **98** (1957), pp. 265–296.
- [18] N. FUSCO, C. SBORDONE: Higher integrability of the gradient of minimizers of functionals with nonstandard growth conditions, *Comm. Pure Appl. Math.* **43** (1990), no. 5, pp. 673–683.
- [19] J. HEINONEN, T. KILPELÄINEN, O. MARTIO: *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Math. Monographs, Clarendon Press, Oxford, 1993.
- [20] S. HWANG: Hölder regularity of solutions of generalized p -Laplacian type parabolic equations, PhD thesis (Paper 12667), Iowa State University, 2012.
- [21] S. HWANG, G. M. LIEBERMAN: Holder continuity of bounded weak solutions to generalized parabolic p -Laplacian equations I: degenerate case, *Electron. J. Diff. Equ.*, **2015** (2015), no. 287, pp. 1–32.
- [22] S. HWANG, G. M. LIEBERMAN: Holder continuity of bounded weak solutions to generalized parabolic p -Laplacian equations II: singular case, *Electron. J. Diff. Equ.*, **2015** (2015), no. 288, pp. 1–24.
- [23] R. KORTE, T. KUUSI, J. SILJANDER: Obstacle problem for nonlinear parabolic equations, *J. Differential Equations* **246** (2009), no.9, pp. 3668–3680.
- [24] R. KORTE, T. KUUSI, M. PARVIAINEN: A connection between a general class of superparabolic functions and supersolutions, *J. Evol. Equ.* **10** (2010), pp. 1–20.
- [25] T. KUUSI: Lower semicontinuity of weak supersolutions to nonlinear parabolic equations, *Differential Integral equations* **22**(11-12) (2009), pp. 1211–1222.
- [26] T. KUUSI, G. MINGIONE, K. NYSTRÖM: Sharp regularity for evolutionary obstacle problems, interpolative geometries and removable sets, *J. Math. Pures Appl.* **101** (2) (2014), 119–151.
- [27] G. M. LIEBERMAN: The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations, *Comm. Part. Diff. Equ.* **16** (1991), pp. 311–361.
- [28] G. M. LIEBERMAN: Boundary and initial regularity for solutions of degenerate parabolic equations, *Nonlinear Anal.* **20** (1993), no. 5, pp. 551–569.
- [29] G. M. LIEBERMAN: Hölder regularity for the gradients of solutions of degenerate parabolic systems, *Nonlinein. Granichnye Zadachi* **16** (2006), pp. 69–85.
- [30] P. LINDQVIST: On the time derivative in an obstacle problem, *Rev. Mat. Iberoam.* **28**(2) (2012), pp. 577–590.
- [31] P. LINDQVIST, J. J. MANFREDI: Viscosity supersolutions of the evolutionary p -Laplace equation, *Differential Integral Equations* **20** (2007), no. 11, pp. 1303–1319.
- [32] P. LINDQVIST, M. PARVIAINEN: Irregular time dependent obstacles, *J. Funct. Anal.* **263** (2012), pp. 2458–2482.
- [33] M. M. PORZIO: L^∞_{loc} -estimates for degenerate and singular parabolic equations, *Nonlinear Anal.* **17** (1991), pp. 1093–1107.

AALTO UNIVERSITY, INSTITUTE OF MATHEMATICS, P.O. BOX 11100, FI-00076 AALTO, FINLAND.
 E-mail address: casimir.lindfors@aalto.fi